$\mathcal{O}$-OPERATORS ON ASSOCIATIVE ALGEBRAS AND ASSOCIATIVE YANG-BAXTER EQUATIONS

CHENGMING BAI, LI GUO, AND XIANG NI

ABSTRACT. We introduce the concept of an extended $\mathcal{O}$-operator that generalizes the well-known concept of a Rota-Baxter operator. We study the associative products coming from these operators and establish the relationship between extended $\mathcal{O}$-operators and the associative Yang-Baxter equation, extended associative Yang-Baxter equation and generalized Yang-Baxter equation.

CONTENTS

1. INTRODUCTION

This paper studies the connection between two concepts in quite different contexts. One is that of a Rota-Baxter operator originated from the probability study of Glenn Baxter [6], influenced by the combinatorial interests of Gian-Carlo Rota [22, 23] and applied broadly in mathematics and physics in recent years [10, 13, 14, 15, 17, 18]. The other concept is that of a solution of the associative Yang-Baxter equations which is an analogue of the classical Yang-Baxter equation in mathematical physics, named after the well-known physicists [7, 28]. A connection between these two objects were first established by Aguiar [1, 2] who

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showed that a solution of the associative Yang-Baxter equation gives rise to a Rota-Baxter operator of weight zero. Such connection has been pursued further in several subsequent papers [3, 5, 11].

We revisit this connection with an alternative approach in order to gain better understanding of the relationship between these two concepts. On one hand we generalize the concept of Rota-Baxter operators to that of $\mathcal{O}$-operators and further to extended $\mathcal{O}$-operators. On the other hand we investigate the operator properties of the associative Yang-Baxter equation motivated by the study in the Lie algebra case [4]. The $\mathcal{O}$-operator is a relative version of the Rota-Baxter operator and, in the context of Lie algebras, was defined by Kupershmidt [20] in the study of Yang-Baxter equations and can be traced back to Bordemann [8] in integrable systems. Through this approach, we show that the operator property of solutions of the associative Yang-Baxter equation is to a large extent characterized by $\mathcal{O}$-operators.

Notations: In this paper, $\mathbf{k}$ denotes a field and is often taken to have characteristic not equal to 2. By an algebra we mean an associative (not necessarily unitary) $\mathbf{k}$-algebra, unless otherwise stated.

1.1. **Rota-Baxter algebras and Yang-Baxter equations.** We recall concepts and relations that motivated our study.

Definition 1.1. Let R be a $\mathbf{k}$-algebra and let $\lambda \in \mathbf{k}$ be given. If a $\mathbf{k}$-linear map $P : R \to R$ satisfies the **Rota-Baxter relation**:

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy), \quad \forall x, y \in R, \tag{1}$$

then P is called a **Rota-Baxter operator of weight** λ and (R, P) is called a **Rota-Baxter algebra of weight** λ**.**

For simplicity, we will only discuss the case of Rota-Baxter operators of weight zero in the introduction.

Note that the relation (1) still makes sense when R is replaced by a $\mathbf{k}$-module with a binary operation. When the binary operation is the Lie bracket and if in addition, the Lie algebra is equipped with a nondegenerate symmetric invariant bilinear form, then a skew-symmetric solution of the **classical Yang-Baxter equation**

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{2}$$

is just a Rota-Baxter operator of weight zero. We refer the reader to [4, 12, 25] for further details.

We consider the following associative analogue of the classical Yang-Baxter equation (2).

Definition 1.2. Let A be a $\mathbf{k}$-algebra. An element $r \in A \otimes A$ is called a **solution of the associative Yang-Baxter equation in** A if it satisfies the relation

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0, \tag{3}$$

called the **associative Yang-Baxter equation (AYBE)**. Here, for $r = \sum\limits_i a_i \otimes b_i \in A \otimes A$, denote

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{4}$$

Both Eq. (3) and another associative analogue of the classical Yang-Baxter equation (2)

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0. \tag{5}$$

were introduced by Aguiar [1, 2, 3]. In fact, Eq. (3) is just Eq. (5) in the opposite algebra [3]. When r is skew-symmetric it is easy to see that Eq. (3) comes from Eq. (5) under the operation $\sigma_{13}(x\otimes y\otimes z) = z\otimes y\otimes x$. While Eq. (5) was emphasized in [1, 2, 3], we will work with Eq. (3) for notational convenience and to be consistent with some of the earlier works on connections with antisymmetric infinitesimal bialgebras [5] and associative D-bialgebras [29].

Theorem 1.3. (Aguiar [2]**)** *Let A be a* $\mathbf{k}$*-algebra. For a solution $r = \sum\limits_i a_i \otimes b_i \in A\otimes A$ of Eq. (5) in A, the map*

$$P : A \to A,\ \ P(x) = \sum_i a_i x b_i,\ \ \forall x \in A,$$

defines a Rota-Baxter operator of weight zero on A.

The theorem is obtained by replacing the tensor symbols in

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = \sum_{i,j} a_i a_j \otimes b_j \otimes b_i - \sum_{i,j} a_i \otimes b_i a_j \otimes b_j + \sum_{i,j} a_j \otimes a_i \otimes b_i b_j = 0,$$

by x and y in A.

1.2. $\mathcal{O}$**-operators.** In this paper, we introduce the concept of an extended $\mathcal{O}$-operator as a generalization of the concept of a Rota-Baxter operator and the associative analogue of an $\mathcal{O}$-operator on a Lie algebra. We then extend the connections of Rota-Baxter algebras with associative Yang-Baxter equations to those of $\mathcal{O}$-operators. This study is motivated by the relationship between $\mathcal{O}$-operator and the classical Yang-Baxter equation in Lie algebras [4, 8, 20]

Let $(A,\cdot)$ be a $\mathbf{k}$-algebra. Let (V,ℓ,r) be an A-bimodule, consisting of a compatible pair of a left A-module (V,ℓ) given by $\ell : A \to \mathrm{End}_{\mathbf{k}}(V)$ and a right A-module (V,r) given by $r : A \to \mathrm{End}_{\mathbf{k}}(V)$ (see Section 2.1 for the precise definition). Fix a $\kappa \in \mathbf{k}$. A pair (α,β) of linear maps $\alpha,\beta : V \to A$ is called an **extended $\mathcal{O}$-operator with modification β of mass κ** if

$$\begin{aligned}&\kappa\ell(\beta(u))v = \kappa u r(\beta(v)),\\ &\alpha(u)\cdot\alpha(v) - \alpha(l(\alpha(u))v + ur(\alpha(v))) = \kappa\beta(u)\cdot\beta(v),\ \ \forall u,v \in V.\end{aligned}$$

When $\beta = 0$ or $\kappa = 0$, we obtain the concept of an **$\mathcal{O}$-operator** α satisfying

$$\alpha(u)\cdot\alpha(v) = \alpha(\ell(\alpha(u))v) + \alpha(ur(\alpha(v))),\ \ \forall u,v \in V. \tag{6}$$

When V is taken to be the A-bimodule (A,L,R) where $L,R : A \to \mathrm{End}_{\mathbf{k}}(A)$ are given by the left and right multiplications, an $\mathcal{O}$-operator $\alpha : V \to A$ of weight zero is just a Rota-Baxter operator of weight zero. To illustrate the close relationship between $\mathcal{O}$-operators and solutions of the AYBE (3), we give the following reformulation of a part of Corollary 3.6. See Section 3 for general cases.

Let $\mathbf{k}$ be a field whose characteristic is not 2. Let A be a $\mathbf{k}$-algebra that we for now assume to have finite dimension over $\mathbf{k}$. Let $\sigma : A\otimes A \to A\otimes A, a\otimes b \mapsto b\otimes a$, be the

switch operator and let $t : \mathrm{Hom}_{\mathbf{k}}(A^*, A) \to \mathrm{Hom}_{\mathbf{k}}(A^*, A)$ be the transpose operator. Then the natural bijection

$$\phi : A \otimes A \to \mathrm{Hom}_{\mathbf{k}}(A^*, \mathbf{k}) \otimes A \to \mathrm{Hom}_{\mathbf{k}}(A^*, A)$$

is compatible with the operators σ and t. Let $\mathrm{Sym}^2(A \otimes A)$ and $\mathrm{Alt}^2(A \otimes A)$ (resp. $\mathrm{Hom}_{\mathbf{k}}(A^*, A)_+$ and $\mathrm{Hom}_{\mathbf{k}}(A^*, A)_-$) be the eigenspaces for the eigenvalues 1 and -1 of σ on $A \otimes A$ (resp. of t on $\mathrm{Hom}_{\mathbf{k}}(A^*, A)$). Thus we have the commutative diagram of bijective linear maps:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\phi"] \arrow[d] & \mathrm{Hom}_{\mathbf{k}}(A^*, A) \arrow[d] \\
\mathrm{Alt}^2(A \otimes A) \oplus \mathrm{Sym}^2(A \otimes A) \arrow[r, "\phi"] & \mathrm{Hom}_{\mathbf{k}}(A^*, A)_- \oplus \mathrm{Hom}_{\mathbf{k}}(A^*, A)_+
\end{tikzcd} \tag{7}$$

preserving the factorizations. Let $\mathrm{Hom}_{bim}(A^*, A)_+$ be the subset of $\mathrm{Hom}_{\mathbf{k}}(A^*, A)_+$ consisting of A-bimodule homomorphisms from A^* to A both of which are equipped with the natural A-bimodule structures. Denote $\mathrm{Sym}^2_{bim}(A \otimes A) := \phi^{-1}(\mathrm{Hom}_{bim}(A^*, A)_+) \subseteq \mathrm{Sym}^2(A \otimes A)$. Then we have (Corollary 3.6)

Theorem 1.4. *An element $r = (r_-, r_+) \in Alt^2(A \otimes A) \oplus Sym^2_{bim}(A \otimes A)$ is a solution of the AYBE (3) if and only if the pair $\phi(r) = (\phi(r)_-, \phi(r)_+) = (\phi(r_-), \phi(r_+))$ is an extended $\mathcal{O}$-operator with modification $\phi(r_+)$ of mass $\kappa = -1$. In particular, when r_+ is zero, an element $r = (r_-, 0) = r_- \in Alt^2(A \otimes A)$ is a solution of the AYBE if and only if the pair $\phi(r) = (\phi(r)_-, 0) = \phi(r_-)$ is an $\mathcal{O}$-operator of weight zero given by Eq. (6) when (V, ℓ, r) is the dual bimodule (A^*, R^*, L^*) of (A, L, R).*

Let $\mathcal{MO}(A^*, A)$ denote the set of extended $\mathcal{O}$-operators (α, β) from A^* to A of mass $\kappa = -1$. Let $\mathcal{O}(A^*, A)$ denote the set of $\mathcal{O}$-operators $\alpha : A^* \to A$ of weight 0. Let $\mathrm{AYB}(A)$ denote the set of solutions of the AYBE (3) in A. Let $\mathrm{SAYB}(A)$ denote the set of skew-symmetric solutions of the AYBE (3) in A. Then Theorem 1.4 means that the bijection in Eq. (7) restricts to bijections in the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Alt}^2(A \otimes A) \oplus \mathrm{Sym}^2_{bim}(A \otimes A) \arrow[r, "\phi"] & \mathrm{Hom}_{\mathbf{k}}(A^*, A)_- \oplus \mathrm{Hom}_{bim}(A^*, A)_+ \\
\mathrm{AYB}(A) \bigcap \Big(\mathrm{Alt}^2(A \otimes A) \oplus \mathrm{Sym}^2_{bim}(A \otimes A)\Big) \arrow[r, "\phi"] \arrow[u, hook] & \mathcal{MO}(A^*, A) \bigcap \Big(\mathrm{Hom}_{\mathbf{k}}(A^*, A)_- \oplus \mathrm{Hom}_{bim}(A^*, A)_+\Big) \arrow[u, hook] \\
\mathrm{SAYB}(A) \arrow[r, "\phi"] \arrow[u, hook] & \mathcal{O}(A^*, A) \bigcap \mathrm{Hom}_{\mathbf{k}}(A^*, A)_- \arrow[u, hook]
\end{tikzcd}$$

1.3. **Layout of the paper.** In Section 2, the concept of an extended $\mathcal{O}$-operator is introduced and its connection with the associativity of certain products is studied. Section 3 establishes the relationship of extended $\mathcal{O}$-operators with associative and extended associative Yang-Baxter equations. Section 4 introduces the concept of the generalized associative Yang-Baxter equation (GAYBE) and considers its relationship with extended $\mathcal{O}$-operators.

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2. $\mathcal{O}$-operators and extended $\mathcal{O}$-operators

We give background notations in Section 2.1 before introducing the concept of an extended $\mathcal{O}$-operator in Section 2.2. We then show in Section 2.3 and 2.4 that extended $\mathcal{O}$-operators can be characterized by the associativity of a multiplication derived from this operator.

2.1. **Bimodules, A-bimodule k-algebras and matched pairs of algebras.** We first recall the concept of a bimodule.

Definition 2.1. *Let $(A, \cdot)$ be a $\mathbf{k}$-algebra.*

(i) *An A-**bimodule** is a $\mathbf{k}$-module V, together with linear maps $\ell, r : A \to \mathrm{End}_{\mathbf{k}}(V)$, such that (V, ℓ) defines a left A-module, (V, r) defines a right A-module and the two module structures on V are compatible in the sense that*

$$(\ell(x)v)r(y) = \ell(x)(vr(y)),\ \forall\ x, y \in A, v \in V.$$

If we want to be more precise, we also denote an A-bimodule V by the triple (V, ℓ, r).

(ii) *A homomorphism between two A-bimodules (V_1, ℓ_1, r_1) and (V_2, ℓ_2, r_2) is a $\mathbf{k}$-linear map $g : V_1 \to V_2$ such that*

$$g(\ell_1(x)v) = \ell_2(x)g(v), \quad g(vr_1(x)) = g(v)r_2(x), \quad \forall\, x \in A, v \in V_1.$$

For a k-algebra A and $x \in A$, define the left and right actions

$$L(x) : A \to A,\ L(x)y = xy\,; \quad R(x) : A \to A,\ yR(x) = yx, \quad y \in A.$$

Further define

$$L = L_A : A \to \mathrm{End}_{\mathbf{k}}(A),\ x \mapsto L(x); \quad R = R_A : A \to \mathrm{End}_{\mathbf{k}}(A),\ x \mapsto R(x), \quad x \in A.$$

Obviously, (A, L, R) is an A-bimodule.

For a $\mathbf{k}$-module V, let $V^* := \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ denote the dual $\mathbf{k}$-module. Denote the usual pairing between V^* and V by

$$\langle\ ,\ \rangle : V^* \times V \to \mathbf{k}, \langle u^*, v\rangle = u^*(v),\ \forall u^* \in V^*, v \in V.$$

Proposition 2.2. ([5]) *Let A be a $\mathbf{k}$-algebra and let (V, ℓ, r) be an A-bimodule. Define the linear maps $\ell^*, r^* : A \to \mathrm{End}_{\mathbf{k}}(V^*)$ by*

$$\langle u^*\ell^*(x), v\rangle = \langle u^*, \ell(x)v\rangle,\ \langle r^*(x)u^*, v\rangle = \langle u^*, vr(x)\rangle,\ \forall x \in A, u^* \in V^*, v \in V, \tag{8}$$

respectively. Then (V^*, r^*, ℓ^*) is an A-bimodule, called the dual bimodule of (V, ℓ, r).

Let (A^*, R^*, L^*) denote the dual A-bimodule of the A-bimodule (A, L, R).

We next extend the concept of a bimodule to that of an A-bimodule algebra by replacing the $\mathbf{k}$-module V by a $\mathbf{k}$-algebra R.

Definition 2.3. Let $(A,\cdot)$ be a $\mathbf{k}$-algebra with multiplication $\cdot$ and let $(R,\circ)$ be a $\mathbf{k}$-algebra with multiplication $\circ$. Let $\ell, r : A \to \mathrm{End}_{\mathbf{k}}(R)$ be two linear maps. We call R (or the triple (R,ℓ,r) or the quadruple $(R,\circ,\ell,r)$) an **A-bimodule k-algebra** if (R,ℓ,r) is an A-bimodule that is compatible with the multiplication $\circ$ on R. More precisely, we have

$$\ell(x\cdot y)v = \ell(x)(\ell(y)v),\ \ell(x)(v\circ w) = (\ell(x)v)\circ w, \tag{9}$$
$$vr(x\cdot y) = (vr(x))r(y),\ (v\circ w)r(x) = v\circ(wr(x)), \tag{10}$$
$$(\ell(x)v)r(y) = \ell(x)(vr(y)),\ (vr(x))\circ w = v\circ(\ell(x)w),\ \forall\, x,y\in A, v,w\in R. \tag{11}$$

Obviously, for any $\mathbf{k}$-algebra $(A,\cdot)$, $(A,\cdot,L,R)$ is an A-bimodule $\mathbf{k}$-algebra. Note that an A-bimodule $\mathbf{k}$-algebra R need not be a left or right A-algebra since we do not assume that $A\cdot 1$ is in the center of R. For example, the A-bimodule $\mathbf{k}$-algebra (A,L,R) is an A-algebra if and only if A is a commutative ring.

The concept of an A-bimodule $\mathbf{k}$-algebra can be further generalized to that of a matched pair introduced in [5].

Definition 2.4. *A* **matched pair** *is a pair of associative algebras $(A,\cdot)$ and $(B,\circ)$ together with linear maps $\ell_A, r_A : A\to \mathrm{End}_{\mathbf{k}}(B)$ and $\ell_B, r_B : B\to\mathrm{End}_{\mathbf{k}}(A)$ such that (B,ℓ_A,r_A) is an A-bimodule and (A,ℓ_B,r_B) is a B-bimodule, satisfying the following conditions:*

$$\ell_A(x)(a\circ b) = \ell_A(x\,r_B(a))b + (\ell_A(x)a)\circ b; \tag{12}$$
$$(a\circ b)\,r_A(x) = a\,r_A(\ell_B(b)x) + a\circ(b\,r_A(x)); \tag{13}$$
$$\ell_B(a)(x\cdot y) = \ell_B(a\,r_A(x))y + (\ell_B(a)x)\cdot y; \tag{14}$$
$$(x\cdot y)\,r_B(a) = x\,r_B(\ell_A(y)a) + x\cdot(y\,r_B(a)); \tag{15}$$
$$\ell_A(\ell_B(a)x)b + (a\,r_A(x))\circ b = a\,r_A(x\,r_B(b)) + a\circ(\ell_A(x)b) = 0; \tag{16}$$
$$\ell_B(\ell_A(x)a)y + (x\,r_B(a))\cdot y = x\,r_B(a\,r_A(y)) + x\cdot(\ell_B(a)y) = 0, \tag{17}$$

for any $x,y\in A, a,b\in B$.

Matching pairs are naturally related to algebraic structures on a direct sum of algebras.

Theorem 2.5. ([5]) *Let $(A,B,\ell_A,r_A,\ell_B,r_B)$ be a matched pair. Then there is an algebra structure on the vector space $A\oplus B$ given by*

$$(x+a)*(y+b) = (x\cdot y+\ell_B(a)y+x\,r_B(b))+(a\circ b+\ell_A(x)b+a\,r_A(y)),\quad \forall x,y\in A, a,b\in B. \tag{18}$$

We denote this associative algebra by $A\bowtie_{\ell_B,r_B}^{\ell_A,r_A} B$ or simply $A\bowtie B$. On the other hand, every associative algebra which is the direct sum (as vector spaces) of two subalgebras can be obtained in this way.

It is clear that an A-bimodule $\mathbf{k}$-algebra (R,ℓ,r) is just a matched pair $(A,R,\ell,r,0,0)$. In fact, in this case, Eq. (12) is the second equation in Eq. (9), Eq. (13) is the second equation in Eq. (10) and Eq. (16) is the second equation in Eq. (11). Equations (14), (15) and (17) hold automatically.

In turn, an A-bimodule (R,ℓ,r) is the special case of an A-bimodule $\mathbf{k}$-algebra when the multiplication $\circ$ on R is the zero product. As a direct corollary of Theorem 2.5, we obtain the following result which is a generalization of the classical result [24] between bimodule structures on V and semi-direct product algebraic structures on $A\oplus V$.

Corollary 2.6. *If $(R, \circ, \ell, r)$ is an A-bimodule* **k***-algebra, then the direct sum $A \oplus R$ of vector spaces is turned into a* **k***-algebra (the semidirect sum) by defining multiplication in $A \oplus R$ by*

$$(x_1, v_1) * (x_2, v_2) = (x_1 \cdot x_2, \ell(x_1)v_2 + v_1 r(x_2) + v_1 \circ v_2), \quad \forall x_1, x_2 \in A, v_1, v_2 \in R.$$

We denote this algebra by $A \ltimes_{\ell,r} R$ or simply $A \ltimes R$.

2.2. **Extended $\mathcal{O}$-operators.** We first define an $\mathcal{O}$-operator before introducing an extended $\mathcal{O}$-operator through an auxiliary operator.

2.2.1. $\mathcal{O}$*-operators.*

Definition 2.7. Let $(A, \cdot)$ be a **k**-algebra and $(R, \circ, \ell, r)$ be an A-bimodule **k**-algebra. A linear map $\alpha : R \to A$ is called an **$\mathcal{O}$-operator of weight $\lambda \in \mathbf{k}$ associated to** $(R, \circ, \ell, r)$ if α satisfies

$$\alpha(u) \cdot \alpha(v) = \alpha(\ell(\alpha(u)v)) + \alpha(ur(\alpha(v))) + \lambda\alpha(u \circ v), \quad \forall u, v \in V. \tag{19}$$

Remark 2.8. Under our assumption that **k** is a field, the non-zero weight can be normalized to weight 1. In fact, for a non-zero weight $\lambda \in \mathbf{k}$, if α is an $\mathcal{O}$-operator of weight λ associated to an A-bimodule **k**-algebra $(R, \circ, \ell, r)$, then α is an $\mathcal{O}$-operator of weight 1 associated to $(R, \lambda\circ, \ell, r)$ and $\frac{1}{\lambda}\alpha$ is an $\mathcal{O}$-operator of weight 1 associated to $(R, \circ, \ell, r)$.

Note that, an A-bimodule (V, ℓ, r) becomes an A-bimodule **k**-algebra when V is equipped with the zero multiplication. Then a linear map $\alpha : V \to A$ is an $\mathcal{O}$-operator (of any weight λ) if

$$\alpha(u) \cdot \alpha(v) = \alpha(\ell(\alpha(u))v) + \alpha(ur(\alpha(v))), \quad \forall u, v \in V.$$

Such a structure appeared independently in [26] under the name of generalized Rota-Baxter operator. Since the weight λ makes no difference in the definition, we just call V an $\mathcal{O}$-operator. This definition recovers the definition in Eq. (6).

Obviously, an $\mathcal{O}$-operator associated to (A, L, R) is just a Rota-Baxter operator on A. An $\mathcal{O}$-operator can be viewed as the relative version of a Rota-Baxter operator in the sense that the domain and range of an $\mathcal{O}$-operator might be different.

2.2.2. *Balanced homomorphisms.* For our purpose of further generalizing the concept of an $\mathcal{O}$-operator, we introduce another concept.

Definition 2.9. Let $(A, \cdot)$ be a **k**-algebra.

(i) Let $\kappa \in \mathbf{k}$ and let (V, ℓ, r) be an A-bimodule. A linear map (resp. an A-bimodule homomorphism) $\beta : V \to A$ is called a **balanced linear map of mass** κ (resp. **balanced A-bimodule homomorphism of mass** κ) if

$$\kappa\ell(\beta(u))v = \kappa ur(\beta(v)), \quad \forall\, u, v \in V. \tag{20}$$

(ii) Let $\kappa, \mu \in \mathbf{k}$ and let $(R, \circ, \ell, r)$ be an A-bimodule **k**-algebra. A linear map (resp. an A-bimodule homomorphism) $\beta : R \to A$ is called a **balanced linear map of mass (κ, μ)** (resp. a **balanced A-bimodule homomorphism of mass** (κ, μ)) if Eq. (20) holds and

$$\mu\ell(\beta(u \circ v))w = \mu ur(\beta(v \circ w)), \quad \forall\, u, v, w \in R. \tag{21}$$

Clearly, if $\kappa = 0$ (resp. $\mu = 0$), then Eq. (20) (resp. Eq. (21)) imposes no restriction. So any A-bimodule homomorphism is balanced of mass $(\kappa, \mu) = (0, 0)$. For a non-zero mass, we have the following examples.

Example 2.10. Let A be a $\mathbf{k}$-algebra.

(i) The identity map $\beta = \mathrm{id} : (A, L, R) \to A$ is a balanced A-bimodule homomorphism (of any mass (κ, μ)).
(ii) Any A-bimodule homomorphism $\beta : (A, L, R) \to A$ is balanced (of any mass (κ, μ)).
(iii) Let $r \in A \otimes A$ be symmetric. If r is regarded as a linear map from (A^*, R^*, L^*) to A is an A-bimodule homomorphism, then r is a balanced A-bimodule homomorphism (of any mass κ). See Lemma 3.2.

2.2.3. *Extended $\mathcal{O}$-operators.* We can now introduce our first main concept in this paper.

Definition 2.11. Let $(A, \cdot)$ be a $\mathbf{k}$-algebra and let $(R, \circ, \ell, r)$ be an A-bimodule $\mathbf{k}$-algebra.

(i) Let $\lambda, \kappa, \mu \in \mathbf{k}$. Fix a balanced A-bimodule homomorphism $\beta : (R, \ell, r) \to A$ of mass (κ, μ). A linear map $\alpha : R \to A$ is called an **extended $\mathcal{O}$-operator of weight λ with modification β of mass** (κ, μ) if

$$\alpha(u) \cdot \alpha(v) - \alpha(\ell(\alpha(u))v + ur(\alpha(v)) + \lambda u \circ v) = \kappa\beta(u) \cdot \beta(v) + \mu\beta(u \circ v), \quad \forall\, u, v \in R. \tag{22}$$

(ii) We also let (α, β) denote an extended $\mathcal{O}$-operator α with modification β.
(iii) When (V, ℓ, r) is an A-bimodule, we regard V as an A-bimodule $\mathbf{k}$-algebra with the zero multiplication. Then λ and μ are irrelevant. We then call the pair (α, β) an **extended $\mathcal{O}$-operator with modification β of mass κ.**

We note that, when the modification β is the zero map (and hence κ and μ are irrelevant), then α is the $\mathcal{O}$-operator defined in Definition 2.7.

2.3. Extended $\mathcal{O}$-operators and associativity.

Let $(A, \cdot)$ be a $\mathbf{k}$-algebra and $(R, \circ, \ell, r)$ be an A-bimodule $\mathbf{k}$-algebra. Let $\delta_\pm : R \to A$ be two linear maps and $\lambda \in \mathbf{k}$. We now consider the associativity of the multiplication

$$u \diamond v := \ell(\delta_+(u))v + ur(\delta_-(v)) + \lambda u \circ v, \quad \forall\, u, v \in R, \tag{23}$$

and several other related multiplications. This will be applied in the Section 4.

Let the characteristic of the field $\mathbf{k}$ be different from 2. Set

$$\alpha := (\delta_+ + \delta_-)/2, \quad \beta := (\delta_+ - \delta_-)/2, \tag{24}$$

called the **symmetrizer** and **antisymmetrizer** of $\delta_\pm$ respectively. Note that $\delta_\pm$ can be recovered from α and β by $\delta_\pm = \alpha \pm \beta$.

Lemma 2.12. *Let $(A, \cdot)$ be a $\mathbf{k}$-algebra and $(R, \circ, \ell, r)$ be an A-bimodule $\mathbf{k}$-algebra. Let $\alpha : R \to A$ be a linear map and let $\lambda \in \mathbf{k}$. Then the operation given by*

$$u *_\alpha v := \ell(\alpha(u))v + ur(\alpha(v)) + \lambda u \circ v, \ \ \forall u, v \in R, \tag{25}$$

is associative if and only if

$$\ell\big(\alpha(u) \cdot \alpha(v) - \alpha(u *_\alpha v)\big)w = ur\big(\alpha(v) \cdot \alpha(w) - \alpha(v *_\alpha w)\big), \tag{26}$$

for all $u, v, w \in R$.

Proof. For any $u, v, w \in R$, we have

$$\begin{aligned}
(u *_\alpha v) *_\alpha w &= \ell(\alpha(u *_\alpha v))w + (u *_\alpha v)r(\alpha(w)) + \lambda(u *_\alpha v) \circ w \\
&= \ell(\alpha(u *_\alpha v))w + (\ell(\alpha(u))v)r(\alpha(w)) + (ur(\alpha(v)))r(\alpha(w)) \\
&\quad +\lambda(u \circ v)r(\alpha(w)) + \lambda(\ell(\alpha(u))v) \circ w + \lambda(ur(\alpha(v))) \circ w + \lambda^2(u \circ v) \circ w.
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
u *_\alpha (v *_\alpha w) &= \ell(\alpha(u))(v *_\alpha w) + ur(\alpha(v *_\alpha w)) + \lambda u \circ (v *_\alpha w) \\
&= \ell(\alpha(u))(\ell(\alpha(v))w) + \ell(\alpha(u))(vr(\alpha(w))) + \lambda\ell(\alpha(u))(v \circ w) \\
&\quad +ur(\alpha(v *_\alpha w)) + \lambda u \circ (\ell(\alpha(v))w) + \lambda u \circ (vr(\alpha(w))) + \lambda^2 u \circ (v \circ w).
\end{aligned} \tag{28}$$

Since $(R, \circ, \ell, r)$ is an A-bimodule **k**-algebra, the second, fourth, fifth, sixth and seventh term on the right hand side of Eq. (27) agrees with the second, sixth, third, fifth and seventh term on the right hand side of Eq. (28) respectively. We further have $(ur(\alpha(v))r(\alpha(w)) = ur(\alpha(v) \circ \alpha(w))$ and $\ell(\alpha(u))(\ell(\alpha(v))w) = \ell(\alpha(u) \circ \alpha(v))w$. We thus have

$$(u *_\alpha v) *_\alpha w - u *_\alpha (v *_\alpha w) = ur\big(\alpha(v) \cdot \alpha(w) - \alpha(v *_\alpha w)\big) - \ell\big(\alpha(u) \cdot \alpha(v) - \alpha(u *_\alpha v)\big)w.$$

Then the lemma follows. □

Corollary 2.13. *Let* **k** *be a field of characteristic not equal to 2. Let $(A, \cdot)$ be a* **k***-algebra and $(R, \circ, \ell, r)$ be an A-bimodule* **k***-algebra. Let $\delta_\pm : R \to A$ be two linear maps and $\lambda \in \boldsymbol{k}$. Let α and β be their symmetrizer and antisymmetrizer defined by Eq. (24). If β is a balanced linear map of mass $\kappa = 1$, namely*

$$\ell(\beta(u))v = ur(\beta(v)), \quad \forall u, v \in R, \tag{29}$$

then the operation $\diamond$ in Eq. (23) defines an associative product on R if and only if α satisfies Eq. (26).

Proof. Since β is balanced, for any $u, v \in R$

$$u \diamond v = \ell(\delta_+(u))v + ur(\delta_-(v)) + \lambda u \circ v = \ell(\alpha(u))v + ur(\alpha(v)) + \lambda u \circ v.$$

Then the conclusion follows from Lemma 2.12. □

Obviously, if α is an $\mathcal{O}$-operator of weight λ associated to an A-bimodule **k**-algebra $(R, \circ, \ell, r)$, then Eq. (26) holds. Thus the operation on R defined by Eq. (23) is associative.

Theorem 2.14. *Let* **k** *have characteristic not equal to 2. Let $(A, \cdot)$ be a* **k***-algebra and $(R, \circ, \ell, r)$ be an A-bimodule* **k***-algebra. Let $\delta_\pm : R \to A$ be two linear maps and $\lambda \in \boldsymbol{k}$. Let α and β be the symmetrizer and antisymmetrizer of $\delta_\pm$.*

(i) *Suppose that β is a balanced linear map of mass (κ, μ) and α satisfies Eq. (22). Then the product $*_\alpha$ is associative.*

(ii) *Suppose β is a balanced A-bimodule homomorphism of mass $(-1, \pm\lambda)$, that is, β satisfies Eq. (20) with $\kappa = -1$, Eq. (21) with $\mu = \pm\lambda$ and*

$$\beta(\ell(x)u) = x \cdot \beta(u), \quad \beta(ur(x)) = \beta(u) \cdot x, \quad \forall x \in A, u \in R. \tag{30}$$

Then α is an extended $\mathcal{O}$-operator of weight λ with modification β of mass $(\kappa, \mu) = (-1, \pm\lambda)$ if and only if $\delta_\pm$ is an $\mathcal{O}$-operator of weight 1 associated to a new A-bimodule **k***-algebra $(R, \circ_\pm, \ell, r)$:*

$$\delta_\pm(u) \cdot \delta_\pm(v) = \delta_\pm(\ell(\delta_\pm(u))v + ur(\delta_\pm(v)) + u \circ_\pm v), \quad \forall u, v \in R, \tag{31}$$

where the associative products $\circ_\pm = \circ_{\lambda,\beta,\pm}$ *on* R *are defined by*

$$u\circ_\pm v = \lambda u \circ v \mp 2\ell(\beta(u))v, \quad \forall u, v \in R. \tag{32}$$

Note that in Item (i) we do not assume that β is an A-bimodule homomorphism. Thus α needs not be an extended $\mathcal{O}$-operator.

Proof. (i) By the relations (20) and (21) and the fact that $(R, \circ, \ell, r)$ is an A-bimodule $\mathbf{k}$-algebra, we obtain

$$\ell(\kappa\beta(u)\cdot\beta(v) + \mu\beta(u\circ v))w = ur(\kappa\beta(v)\cdot\beta(w) + \mu\beta(v\circ w)),\ \ \forall u, v, w \in R.$$

Since Eq. (22) also holds, the above equation implies Eq. (26) and hence the associativity of $*_\alpha$ by Lemma 2.12.

(ii) First we prove that the operations $\circ_\pm$ defined by Eq. (32) make $(R, \circ_\pm, \ell, r)$ into an A-bimodule $\mathbf{k}$-algebra. In fact, for any $u, v, w \in R$,

$$\begin{aligned}
(u\circ_\pm v)\circ_\pm w &= (\lambda u\circ v \mp 2l(\beta(u))v)\circ_\pm w\\
&= \lambda^2(u\circ v)\circ w \mp 2\lambda(\ell(\beta(u))v)\circ w \mp 2\lambda\ell(\beta(u\circ v))w + 4\ell(\beta(\ell(\beta(u))v))w\\
&= \lambda^2 u\circ(v\circ w) \mp 2\lambda u\circ(\ell(\beta(v))w) \mp 2\lambda\ell(\beta(u))(v\circ w) + 4\ell(\beta(u))(\ell(\beta(v))w)\\
&= u\circ_\pm(\lambda v\circ w \mp 2\ell(\beta(v))w) = u\circ_\pm(v\circ_\pm w),
\end{aligned}$$

where the third equality follows since each term on one side of the equation equals to the corresponding term on the other side by our assumptions on β and the fact that $(R, \circ, \ell, r)$ is an A-bimodule $\mathbf{k}$-algebra. On the other hand, for any $x \in A$,

$$\begin{aligned}
\ell(x)(u\circ_\pm v) &= \ell(x)(\lambda u\circ v \mp 2\ell(\beta(u))v)\\
&= \lambda(\ell(x)u)\circ v \mp 2\ell(x\cdot\beta(u))v \quad \text{(by Eq. (9))}\\
&= \lambda(\ell(x)u)\circ v \mp 2\ell(\beta(\ell(x)u))v \quad \text{(by Eq. (30))}\\
&= (\ell(x)u)\circ_\pm v.
\end{aligned}$$

By the same argument, we have $(u\circ_\pm v)r(x) = u\circ_\pm(vr(x))$. Moreover,

$$\begin{aligned}
(ur(x))\circ_\pm v &= \lambda(ur(x))\circ v \mp 2\ell(\beta(ur(x)))v\\
&= \lambda u\circ(\ell(x)v) \mp 2\ell(\beta(u))(\ell(x)v) \quad \text{(by Eq. (30), Eq. (9) and Eq. (11))}\\
&= u\circ_\pm(\ell(x)v).
\end{aligned}$$

The other axioms in the Definition 2.3 of an A-bimodule $\mathbf{k}$-algebra do not depend on the product of R. Thus $(R, \circ_\pm, \ell, r)$ equipped with the product $\circ_\pm$ is an A-bimodule $\mathbf{k}$-algebra. Moreover,

$$\begin{aligned}
&(\alpha\pm\beta)(u)\cdot(\alpha\pm\beta)(v) - (\alpha\pm\beta)(\ell((\alpha\pm\beta)(u))v + ur((\alpha\pm\beta)(v)) + u\circ_\pm v)\\
= {}&\alpha(u)\cdot\alpha(v) + \beta(u)\cdot\beta(v) - \alpha(\ell(\alpha(u))v + ur(\alpha(v)) + \lambda u\circ v) \mp \lambda\beta(u\circ v)\\
&\pm(\beta(u)\cdot\alpha(v) - \beta(ur(\alpha(v))) + \alpha(u)\cdot\beta(v) - \beta(\ell(\alpha(u))v)) \quad \text{(by Eq. (29))}\\
= {}&\alpha(u)\cdot\alpha(v) + \beta(u)\cdot\beta(v) - \alpha(\ell(\alpha(u))v + ur(\alpha(v)) + \lambda u\circ v) \mp \lambda\beta(u\circ v) \text{ (by Eq. (30))}.
\end{aligned}$$

So α and β satisfy Eq. (22) with $\kappa = -1$ and $\mu = \pm\lambda$ if and only if $\delta_\pm$ is an $\mathcal{O}$-operator of weight 1 from $(R, \circ_\pm, \ell, r)$ to A. □

We close this section with an obvious corollary of Theorem 2.14.

Corollary 2.15. *Let A be a $\mathbf{k}$-algebra and (V,ℓ,r) be an A-bimodule. Let $\alpha,\beta: V\to A$ be two linear maps such that β is a balanced A-bimodule homomorphism. Then α is an extended $\mathcal{O}$-operator with modification β of mass $\kappa=-1$ if and only if $\alpha\pm\beta$ is an $\mathcal{O}$-operator of weight 1 associated to an A-bimodule $\mathbf{k}$-algebra $(V,\star_\pm,\ell,r)$, i.e.,*

$$(\alpha\pm\beta)(u)\cdot(\alpha\pm\beta)(v)=(\alpha\pm\beta)(\ell((\alpha\pm\beta)(u))v+ur((\alpha\pm\beta)(v))+u\star_\pm v),\quad \forall u,v\in V,$$

where the associative algebra products $\star_\pm$ on V are defined by

$$u\star_\pm v=\mp 2\ell(\beta(u))v,\quad \forall u,v\in V.$$

Proof. The corollary follows by taking $R=V$ in Theorem 2.14. (ii) with the zero multiplication. □

2.4. **The case of $\mathcal{O}$-operators and Rota-Baxter operators.** Let $(A,\cdot)$ be a $\mathbf{k}$-algebra. Then $(A,\cdot,L,R)$ is an A-bimodule $\mathbf{k}$-algebra. Theorem 2.14 can be easily restated in this case. But we are mostly interested in the case of $\mu=0$ when Eq. (22) takes the form

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y)+\lambda x\cdot y)=\kappa\beta(x)\cdot\beta(y),\quad \forall x,y\in A. \tag{33}$$

We list the following special cases for later reference. When $\lambda=0$, Eq. (33) gives

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y))=\kappa\beta(x)\cdot\beta(y),\quad \forall x,y\in A. \tag{34}$$

If in addition, $\beta=\mathrm{id}$, then Eq. (34) gives

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y))=\kappa x\cdot y,\ \ \forall x,y\in A. \tag{35}$$

When furthermore $\kappa=-1$, Eq. (35) becomes

$$\alpha(x)\cdot\alpha(y)-\alpha(\alpha(x)\cdot y+x\cdot\alpha(y))=-x\cdot y,\ \ \forall x,y\in A. \tag{36}$$

By the proof of Lemma 2.12 and Theorem 2.14, we reach the following conclusion.

Corollary 2.16. *Let $(A,\cdot)$ be a $\mathbf{k}$-algebra. Let $\alpha,\beta: A\to A$ be two linear maps and $\lambda\in\boldsymbol{k}$.*

(i) *For any $\kappa\in\boldsymbol{k}$, let β be balanced of mass $(\kappa,0)$ and let α be an extended $\mathcal{O}$-operator of weight λ with modification β of mass $(\kappa,\mu)=(\kappa,0)$, namely α satisfies Eq. (33). Then the product $*_\alpha$ on A is associative.*

(ii) *If β is an A-bimodule homomorphism, then α and β satisfy Eq. (34) for $\kappa=-1$ if and only if $r_\pm=\alpha\pm\beta$ is an $\mathcal{O}$-operator of weight 1 associated to a new A-bimodule $\mathbf{k}$-algebra $(A,\star_\pm,L,R)$:*

$$r_\pm(x)\cdot r_\pm(y)=r_\pm(r_\pm(x)\cdot y+x\cdot r_\pm(y)+x\star_\pm y),\quad \forall x,y\in A,$$

where the associative products $\star_\pm$ on A are defined by

$$x\star_\pm y=\mp 2\beta(x)\cdot y,\quad \forall x,y\in A.$$

Remark 2.17. With the notations as above, if $\beta: A\to A$ is an A-bimodule homomorphism, then β is balanced of mass $(\kappa,0)$. Moreover, in this case, β is an **averaging operator** [2, 23], namely,

$$\beta(x)\cdot\beta(y)=\beta(x\cdot\beta(y))=\beta(\beta(x)\cdot y),\quad \forall x,y\in A, \tag{37}$$

and it is also a **Nijenhuis operator** [9, 12], namely,

$$\beta(x)\cdot\beta(y)+\beta^2(x\cdot y)=\beta(x\cdot\beta(y)+\beta(x)\cdot y),\quad \forall x,y\in A.$$

Let $(A, \cdot)$ be a $\mathbf{k}$-algebra and let $(A, \cdot, L, R)$ be the corresponding A-bimodule $\mathbf{k}$-algebra. In this case, it is obvious that $\beta = \mathrm{id}$ satisfies the conditions of Theorem 2.14 and Eq. (22) takes the form

$$\alpha(x)\cdot\alpha(y) - \alpha(\alpha(x)\cdot y + x\cdot\alpha(y) + \lambda x\cdot y) = \hat{\kappa} x\cdot y, \quad \forall x, y \in A, \tag{38}$$

where $\hat{\kappa} = \kappa + \mu$. Thus we have the following consequence of Theorem 2.14.

Corollary 2.18. *Let $\hat{\kappa} = -1 \pm \lambda$. Then $\alpha : A \to A$ satisfies Eq. (38) if and only if $\alpha \pm 1$ is a Rota-Baxter operator of weight $\lambda \mp 2$.*

When $\lambda = 0$, this fact can be found in [11]. As noted there, the Lie algebraic version of Eq. (38) in this case, namely Eq. (36), is the operator form of the modified classical Yang-Baxter equation [25].

3. Extended $\mathcal{O}$-operators and AYBE

In this section we study the relationship between extended $\mathcal{O}$-operators and associative Yang-Baxter equations. We start with introducing various concepts of the associative Yang-Baxter equation (AYBE) in Section 3.1. We then establish connections between $\mathcal{O}$-operators in different generalities and solutions of these variations of AYBE in different algebras. The relationship between $\mathcal{O}$-operators on a $\mathbf{k}$-algebra A and solutions of AYBE in A is considered in Section 3.2. We consider the special case of Frobenius algebras in Section 3.3. We finally consider in Section 3.4 the relationship between an extended $\mathcal{O}$-operator and solutions of AYBE and extended AYBE in an extension algebra of A.

3.1. **Extended associative Yang-Baxter equations.** We define variations of the associative Yang-Baxter equation to be satisfied by two tensors from an algebra. We then study the linear maps from these two tensors in preparation for the relationship between $\mathcal{O}$-operators and solutions of these associative Yang-Baxter equations.

Let A be a $\mathbf{k}$-algebra. Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. We continue to use the notations r_{12}, r_{13} and r_{23} defined in Eq. (4). We similarly define

$$r_{21} = \sum_i b_i \otimes a_i \otimes 1, \quad r_{31} = \sum_i b_i \otimes 1 \otimes a_i, \quad r_{32} = \sum_i 1 \otimes b_i \otimes a_i.$$

Equip $A \otimes A \otimes A$ with the product of the tensor algebra. In particular,

$$(a_1 \otimes a_2 \otimes a_3)(b_1 \otimes b_2 \otimes b_3) = (a_1 b_1) \otimes (a_2 b_2) \otimes (a_3 b_3), \quad \forall\, a_i, b_i \in A, i = 1, 2, 3.$$

Definition 3.1. Fix $\varepsilon \in \mathbf{k}$.

(i) The equation

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = \varepsilon(r_{13} + r_{31})(r_{23} + r_{32}) \tag{39}$$

is called the **extended associative Yang-Baxter equation of mass** ε (or ε-**EAYBE** in short).

(ii) Let A be a $\mathbf{k}$-algebra. An element $r \in A \otimes A$ is called a **solution of the** ε**-EAYBE in** A if it satisfies Eq. (39).

When $\varepsilon = 0$ or r is skew-symmetric in the sense that $\sigma(r) = -r$ for the switch operator $\sigma : A \otimes A \to A \otimes A$ (and hence $r_{13} = -r_{31}$), then the ε-EAYBE is the same as the AYBE in Eq. (3):

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0. \tag{40}$$

Let A be a $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension. For $r \in A \otimes A$, define a linear map $F_r : A^* \to A$ by

$$\langle v, F_r(u)\rangle = \langle u \otimes v, r\rangle, \ \ \forall u, v \in A^*. \tag{41}$$

This defines a bijective linear map $F : A \otimes A \to \mathrm{Hom}_{\mathbf{k}}(A^*, A)$ and thus allows us to identify r with F_r which we still denote by r for simplicity of notations. Similarly define a linear map $r^t : A^* \to A$ by

$$\langle u, r^t(v)\rangle = \langle r, u \otimes v\rangle. \tag{42}$$

Obviously r is symmetric or skew-symmetric in $A \otimes A$ if and only if, as a linear map, $r = r^t$ or $r = -r^t$ respectively. Suppose that the characteristic of $\mathbf{k}$ is not 2 and define

$$\alpha = \alpha_r = (r - r^t)/2, \quad \beta = \beta_r = (r + r^t)/2, \tag{43}$$

called the **skew-symmetric part** and the **symmetric part** of r respectively. Then $r = \alpha + \beta$ and $r^t = -\alpha + \beta$.

Lemma 3.2. *Let $(A, \cdot)$ be a $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension. Let $s \in A \otimes A$ be symmetric. Then the following conditions are equivalent.*

(i) *s is* **invariant***, i.e.,*

$$(\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})s = 0, \ \ \forall x \in A. \tag{44}$$

(ii) *s regarded as a linear map from (A^*, R^*, L^*) to A is balanced, i.e.,*

$$R^*(s(a^*))b^* = a^* L^*(s(b^*)), \quad \forall a^*, b^* \in A^*. \tag{45}$$

(iii) *s regarded as a linear map from (A^*, R^*, L^*) to A is an A-bimodule homomorphism, i.e.,*

$$s(R^*(x)a^*) = x \cdot s(a^*), \quad s(a^* L^*(x)) = s(a^*) \cdot x, \quad \forall x \in A, a^* \in A^*. \tag{46}$$

Proof. "(i)$\Leftrightarrow$(ii)". Since $s \in A \otimes A$ is symmetric, for any $x \in A, a^*, b^* \in A^*$,

$$\begin{aligned}
\langle (\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})s, a^* \otimes b^*\rangle &= \langle s, a^* \otimes L^*(x)b^*\rangle - \langle s, R^*(x)a^* \otimes b^*\rangle \\
&= \langle x \cdot s(a^*), b^*\rangle - \langle a^*, s(b^*) \cdot x\rangle \\
&= \langle R^*(s(a^*))b^* - a^* L^*(s(b^*)), x\rangle.
\end{aligned}$$

So s is invariant if and only if s regarded as a linear map from (A^*, R^*, L^*) to A is balanced.

"(i)$\Leftrightarrow$(iii)". For any $x \in A, a^*, b^* \in A^*$,

$$\begin{aligned}
\langle (\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})s, a^* \otimes b^*\rangle &= \langle s, a^* \otimes L^*(x)b^*\rangle - \langle s, R^*(x)a^* \otimes b^*\rangle \\
&= \langle x \cdot s(a^*) - s(R^*(x)a^*), b^*\rangle \\
\langle (\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})s, a^* \otimes b^*\rangle &= \langle s, a^* \otimes L^*(x)b^*\rangle - \langle s, R^*(x)a^* \otimes b^*\rangle \\
&= \langle s(L^*(x)b^*) - s(b^*) \cdot x, a^*\rangle,
\end{aligned}$$

by the symmetry of $s \in A \otimes A$. So s is invariant if and only if s regarded as a linear map from (A^*, R^*, L^*) to A is an A-bimodule homomorphism. $\square$

Remark 3.3. The invariant condition in Item (i) also arises in the construction of a coboundary antisymmetric infinitesimal bialgebra in the sense of [5] (see also [21]).

3.2. **Extended $\mathcal{O}$-operators and EAYBE.** We first state the following special case of Corollary 2.15.

Corollary 3.4. *Let* $\mathbf{k}$ *be a field of characteristic not equal to 2. Let* A *be a* $\mathbf{k}$*-algebra with finite* $\mathbf{k}$*-dimension and* $r \in A \otimes A$*. Let* α, β *be defined by Eq. (43). Suppose* β *is a balanced* A*-bimodule homomorphism. The following two statements are equivalent.*

(i) *The map* α *is an extended* $\mathcal{O}$*-operator with modification* β *of mass* -1*:*

$$\alpha(a^*)\cdot\alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^*L^*(\alpha(b^*))) = -\beta(a^*)\cdot\beta(b^*), \quad \forall a^*, b^* \in A^*. \tag{47}$$

(ii) *The map* r *(resp.* $-r^t$*) is an* $\mathcal{O}$*-operator of weight 1 associated to a new* A*-bimodule* $\mathbf{k}$*-algebra* $(A^*, \circ_+, R^*, L^*)$ *(resp.* $(A^*, \circ_-, R^*, L^*)$ *):*

$$r(a^*)\cdot r(b^*) = r(R^*(r(a^*))b^* + a^*L^*(r(b^*)) + a^*\circ_+ b^*), \quad \forall a^*, b^* \in A^*, \tag{48}$$

(resp.

$$(-r^t)(a^*)\cdot(-r^t)(b^*) = (-r^t)(R^*((-r^t)(a^*))b^* + a^*L^*((-r^t)(b^*)) + a^*\circ_- b^*), \tag{49}$$

$\forall a^*, b^* \in A^*$*), where the associative algebra products* $\circ_\pm$ *on* A^* *are defined by*

$$a^*\circ_\pm b^* = \mp 2R^*(\beta(a^*))b^*, \quad \forall a^*, b^* \in A^*. \tag{50}$$

In the theory of integrable systems [19, 25], **modified classical Yang-Baxter equation** is usually referred to (the Lie algebraic version of) Eq. (36) and Eq. (47).

The following theorem establishes a close relationship between extended $\mathcal{O}$-operators on a $\mathbf{k}$-algebra A and solutions of the AYBE in A.

Theorem 3.5. *Let* $\mathbf{k}$ *be a field of characteristic not equal to 2. Let* A *be a* $\mathbf{k}$*-algebra with finite* $\mathbf{k}$*-dimension and let* $r \in A \otimes A$ *which is identified as a linear map from* A^* *to* A*.*

(i) *Then* r *is a solution of the AYBE in* A *if and only if* r *satisfies*

$$r(a^*)\cdot r(b^*) = r(R^*(r(a^*))b^* - a^*L^*(r^t(b^*))), \;\; \forall a^*, b^* \in A^*. \tag{51}$$

(ii) *Define* α *and* β *by Eq. (43). Suppose that the symmetric part* β *of* r *is invariant. Then* r *is a solution of EAYBE of mass* $\frac{\kappa+1}{4}$*:*

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = \frac{\kappa+1}{4}(r_{13}+r_{31})(r_{23}+r_{32})$$

if and only if α *is an extended* $\mathcal{O}$*-operator with modification* β *of mass* κ*:*

$$\alpha(a^*)\cdot\alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^*L^*(\alpha(b^*))) = \kappa\beta(a^*)\cdot\beta(b^*), \quad \forall a^*, b^* \in A^*.$$

Proof. (i) Denote $r = \sum_{i,j} u_i \otimes v_j$. For any $a^*, b^*, c^* \in A^*$, we have

$$\begin{aligned}
\langle r_{12} \cdot r_{13}, a^* \otimes b^* \otimes c^* \rangle &= \sum_{i,j} \langle u_i \cdot u_j, a^* \rangle \langle v_i, b^* \rangle \langle v_j, c^* \rangle = \sum_j \langle r^t(b^*) \cdot u_j, a^* \rangle \langle v_j, b^* \rangle \\
&= \langle r(a^* L^*(r^t(b^*))), c^* \rangle, \\
\langle r_{13} \cdot r_{23}, a^* \otimes b^* \otimes c^* \rangle &= \sum_{i,j} \langle u_i, a^* \rangle \langle u_j, b^* \rangle \langle v_i \cdot v_j, c^* \rangle = \sum_j \langle u_j, b^* \rangle \langle r(a^*) \cdot v_j, c^* \rangle \\
&= \langle r(a^*) \cdot r(b^*), c^* \rangle, \\
\langle -r_{23} \cdot r_{12}, a^* \otimes b^* \otimes c^* \rangle &= -\sum_{i,j} \langle u_i, a^* \rangle \langle u_j \cdot v_i, b^* \rangle \langle v_j, c^* \rangle = -\sum_j \langle u_j \cdot r(a^*), b^* \rangle \langle v_j, c^* \rangle \\
&= \langle -r(R^*(r(a^*))b^*), c^* \rangle.
\end{aligned}$$

Therefore r is a solution of the AYBE in A if and only if r satisfies Eq. (51).

(ii) By the proof of Item (i), we see that, for any $a^*, b^*, c^* \in A^*$,

$$\begin{aligned}
&\langle \alpha(a^*) \cdot \alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^* L^*(\alpha(b^*))) - \kappa \beta(a^*) \cdot \beta(b^*), c^* \rangle \\
= \; &\langle \alpha(a^*) \cdot \alpha(b^*) - \alpha(R^*(\alpha(a^*))b^* + a^* L^*(\alpha(b^*))) + \beta(a^*) \cdot \beta(b^*) - (\kappa+1)\beta(a^*) \cdot \beta(b^*), c^* \rangle \\
= \; &\langle r_{12} \cdot r_{13} + r_{13} \cdot r_{23} - r_{23} \cdot r_{12}, a^* \otimes b^* \otimes c^* \rangle - (\kappa+1) \langle \beta_{13} \cdot \beta_{23}, a^* \otimes b^* \otimes c^* \rangle \\
= \; &\langle r_{12} \cdot r_{13} + r_{13} \cdot r_{23} - r_{23} \cdot r_{12} - (\kappa+1) \frac{r_{13} + r_{31}}{2} \cdot \frac{r_{23} + r_{32}}{2}, a^* \otimes b^* \otimes c^* \rangle.
\end{aligned}$$

So r is a solution of the EAYBE of mass $(\kappa+1)/4$ if and only if α is an extended $\mathcal{O}$-operator with modification β of mass κ. $\square$

In the case when $\kappa = -1$, we have

Corollary 3.6. *Let* $\mathbf{k}$ *be a field of characteristic not equal to 2. Let* A *be a* $\mathbf{k}$*-algebra with finite* $\mathbf{k}$*-dimension and let* $r \in A \otimes A$*. Define* α *and* β *by Eq. (43).*

(i) *If* β *is invariant, then the following conditions are equivalent.*
   (a) r *is a solution of the AYBE in* A*.*
   (b) r *(resp.* $-r^t$*) satisfies Eq. (48) (resp. Eq. (49)), i.e.,* r *(resp.* $-r^t$*) is an* $\mathcal{O}$*-operator of weight 1 associated to the* A*-bimodule* $\mathbf{k}$*-algebra* $(A^*, \circ_+, R^*, L^*)$ *(resp.* $(A^*, \circ_-, R^*, L^*)$*), where* A^* *is equipped with the associative algebra structure* $\circ_+$ *(resp.* $\circ_-$*) defined by Eq. (50).*
   (c) α *is an extended* $\mathcal{O}$*-operator with modification* β *of mass* -1*.*
   (d) *For any* $a^*, b^* \in A^*$*,*

(52)
$$(\alpha \pm \beta)(a^* * b^*) = (\alpha \pm \beta)(a^*) \cdot (\alpha \pm \beta)(b^*),$$

where

$$a^* * b^* = R^*(r(a^*))b^* - a^* L^*(r^t(b^*)), \;\; \forall a^*, b^* \in A^*.$$

(ii) *When* r *is skew-symmetric, then* r *is a solution of the AYBE in* A *if and only if* $r : A^* \to A$ *is an* $\mathcal{O}$*-operator of weight zero.*

Proof. If the symmetric part β of r is invariant, then by Lemma 3.2, for any $a^*, b^* \in A^*$, we have

$$\begin{aligned}
& r(a^*)\cdot r(b^*) - r(R^*(r(a^*))b^* - a^*L^*(r^t(b^*))) \\
= \; & r(a^*)\cdot r(b^*) - r(R^*(r(a^*))b^* + a^*L^*(r(b^*)) - 2a^*L^*(\beta(b^*))) \\
= \; & r(a^*)\cdot r(b^*) - r(R^*(r(a^*))b^* + a^*L^*(r(b^*)) + a^*\circ_+ b^*),
\end{aligned}$$

where the product $\circ_+$ is defined by Eq. (50). So by Corollary 3.4, r is a solution of the AYBE if and only if Item (ib) or (ic) holds. Moreover, since for any $a^*, b^* \in A^*$,

$$\begin{aligned}
R^*(r(a^*))b^* - a^*L^*(r^t(b^*)) &= R^*(r(a^*))b^* + a^*L^*(r(b^*)) + a^*\circ_+ b^* \\
&= R^*((-r^t)(a^*))b^* + a^*L^*((-r^t)(b^*)) + a^*\circ_- b^*,
\end{aligned}$$

Eq. (52) is just a reformulation of Eq. (48) and Eq. (49). So r is a solution of the AYBE if and only if Item (id) holds.

(ii) This is the special case of Item (i) when $\beta = 0$. □

3.3. **$\mathcal{O}$-operators and AYBE on Frobenius algebras.** In this section we consider the relationship between $\mathcal{O}$-operators and solutions of the AYBE on Frobenius algebras.

Definition 3.7. (i) *Let A be a $\mathbf{k}$-algebra and let $B(\ ,\ ) : A\otimes A \to \mathbf{k}$ be a nondegenerate bilinear form. Let $\varphi : A \to A^*$ denote the induced injective linear map defined by*

$$B(x, y) = \langle \varphi(x), y\rangle, \quad \forall x, y \in A. \tag{53}$$

(ii) *A* **Frobenius k-algebra** *is a $\mathbf{k}$-algebra $(A,\cdot)$ together with a nondegenerate bilinear form $B(\ ,\ ) : A \otimes A \to \mathbf{k}$ that is invariant in the sense that*

$$B(x \cdot y, z) = B(x, y \cdot z),\ \forall x, y, z \in A.$$

We use $(A, \cdot, B)$ to denote a Frobenius $\mathbf{k}$-algebra.

(iii) *A Frobenius $\mathbf{k}$-algebra is called* **symmetric** *if*

$$B(x, y) = B(y, x),\ \forall x, y \in A.$$

(iv) *A linear map $\beta : A \to A$ is called* **self-adjoint** *(resp.* **skew-adjoint***) with respect to a bilinear form B if for any $x, y \in A$, we have $B(\beta(x), y) = B(x, \beta(y))$ (resp. $B(\beta(x), y) = -B(x, \beta(y))$).*

The study of Frobenius algebras was originated from the work [16] of Frobenius and has found applications in broad areas of mathematics and physics. See [27, 5] for further details.

It is easy to get the following result.

Proposition 3.8. *([27]) Let A be a symmetric Frobenius $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension. Then the A-bimodule (A, L, R) is isomorphic to the A-bimodule (A^*, R^*, L^*).*

The following statement gives a class of symmetric Frobenius algebras from symmetric, invariant tensors.

Corollary 3.9. *Let $(A,\cdot)$ be a $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension. Let $s \in A \otimes A$ be symmetric and invariant. Suppose that s regarded as a linear map from $A^* \to A$ is invertible. Then $s^{-1} : A \to A^*$ regraded as a bilinear form $B(\ ,\ ) : A \otimes A \to \boldsymbol{k}$ on A through Eq. (53) for $\varphi = s^{-1}$ is symmetric, nondegenerate and invariant. Thus $(A, \cdot, B)$ is a symmetric Frobenius algebra.*

Proof. Since s is symmetric and s regarded as a linear map from A^* to A is invertible, $B(\ ,\ )$ is symmetric and nondegenerate. On the other hand, since s is invariant, by Lemma 3.2 Eq. (45) holds. Thus, for any $x, y, z \in A$ and $a^* = s^{-1}(x), b^* = s^{-1}(y), c^* = s^{-1}(z)$, we have

$$\begin{aligned} B(x \cdot y, z) &= \langle c^*, s(a^*) \cdot s(b^*)\rangle = \langle c^* L^*(s(a^*)), b^*\rangle \\ &= \langle R^*(s(c^*))a^*, b^*\rangle = \langle a^*, s(b^*) \cdot s(c^*)\rangle = B(x, y \cdot z), \end{aligned}$$

i.e., $B(\ ,\ )$ is invariant. So the conclusion follows. □

Lemma 3.10. *Let $(A, \cdot, B)$ be a symmetric Frobenius $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension. Suppose that $\beta : A \to A$ is an endomorphism of A which is self-adjoint with respect to B. Then $\tilde{\beta} = \beta\varphi^{-1} : A^* \to A$ regarded as an element of $A \otimes A$ is symmetric, where $\varphi : A \to A^*$ is defined by Eq. (53). Moreover, β is a balanced A-bimodule homomorphism if and only if $\tilde{\beta}$ is a balanced A-bimodule homomorphism.*

Proof. For any $a^*, b^* \in A^*$, set $x = \phi^{-1}(a^*), y = \phi^{-1}(b^*)$. Since β is self-adjoint with respect to B, we see that

$$\langle \tilde{\beta}(a^*), b^*\rangle = \langle \beta(x), \varphi(y)\rangle = B(\beta(x), y) = B(x, \beta(y)) = \langle \varphi(x), \beta(y)\rangle = \langle a^*, \tilde{\beta}(b^*)\rangle.$$

Therefore, $\tilde{\beta}$ regarded as an element of $A \otimes A$ is symmetric. Moreover, since B is symmetric and invariant and β is self-adjoint with respect to B, for any $a^*, b^* \in A^*, z \in A$ and $x = \varphi^{-1}(a^*), y = \varphi^{-1}(b^*)$, we have

$$\begin{aligned} \langle R^*(\tilde{\beta}(a^*))b^*, z\rangle &= \langle R^*(\beta(x))\varphi(y), z\rangle = B(y, z \cdot \beta(x)) \\ \langle a^* L^*(\tilde{\beta}(b^*)), z\rangle &= \langle \varphi(x) L^*(\beta(y)), z\rangle = B(x, \beta(y) \cdot z) = B(\beta(y), z \cdot x) = B(y, \beta(z \cdot x)). \end{aligned}$$

Thus $\tilde{\beta}$ satisfies Eq. (45) if and only if $\beta(z \cdot x) = z \cdot \beta(x)$, for any $x, z \in A$. On the other hand,

$$\begin{aligned} \langle R^*(\tilde{\beta}(a^*))b^*, z\rangle &= \langle R^*(\beta(x))\varphi(y), z\rangle = B(y, z \cdot \beta(x)) = B(\beta(x), y \cdot z) = B(x, \beta(y \cdot z)), \\ \langle a^* L^*(\tilde{\beta}(b^*)), z\rangle &= \langle \varphi(x) L^*(\beta(y)), z\rangle = B(x, \beta(y) \cdot z). \end{aligned}$$

Therefore, $\tilde{\beta}$ satisfies Eq. (45) if and only if $\beta(y \cdot z) = \beta(y) \cdot z$, for any $y, z \in A$. Hence β is an A-bimodule homomorphism if and only if $\tilde{\beta}$ is an A-bimodule homomorphism. Furthermore, the equivalence between Eq. (45) and Eq. (46) follows from Lemma 3.2. □

If $\beta = \mathrm{id}$, then the above lemma says that $\varphi^{-1} : A^* \to A$ is a balanced A-bimodule homomorphism.

Corollary 3.11. *Let $(A, \cdot, B)$ be a symmetric Frobenius $\mathbf{k}$-algebra of finite $\mathbf{k}$-dimension and let $\varphi : A \to A^*$ be the linear map defined by Eq. (53). Suppose $\beta \in A \otimes A$ is symmetric. Then β regarded as a linear map from (A^*, R^*, L^*) to A is a balanced A-bimodule homomorphism if and only if $\hat{\beta} = \beta\varphi : A \to A$ is a balanced A-bimodule homomorphism.*

Proof. Since $\beta \in A \otimes A$ is symmetric, for any $x, y \in A$,

$$\begin{aligned} & \langle \beta, \varphi(x) \otimes \varphi(y)\rangle = \langle \beta, \varphi(y) \otimes \varphi(x)\rangle \\ \Leftrightarrow\ & \langle \beta(\varphi(x)), \varphi(y)\rangle = \langle \beta(\varphi(y)), \varphi(x)\rangle \\ \Leftrightarrow\ & B(\hat{\beta}(x), y) = B(\hat{\beta}(y), x). \end{aligned}$$

Thus $\hat{\beta} = \beta\varphi$ is self-adjoint with respect to $B(\ ,\ )$. So the conclusion follows from Lemma 3.10. □

Theorem 3.12. *Let $\mathbf{k}$ be a field of characteristic not equal to 2. Let $(A, \cdot, B)$ be a symmetric Frobenius algebra of finite $\mathbf{k}$-dimension. Suppose that α and β are two endomorphisms of A and β is self-adjoint with respect to B.*

(i) *α is an extended $\mathcal{O}$-operator with modification β of mass κ if and only if $\tilde{\alpha} := \alpha\circ\varphi^{-1} : A^* \to A$ is an extended $\mathcal{O}$-operator with modification $\tilde{\beta} := \beta\circ\varphi^{-1} : A^* \to A$ of mass κ, where the linear map $\varphi : A \to A^*$ is defined by Eq. (53).*

(ii) *Suppose that in addition, α is skew-adjoint with respect to B. Then $\tilde{\alpha}$ regarded as an element of $A\otimes A$ is skew-symmetric and we have*
  (a) *$r_\pm = \tilde{\alpha}\pm\tilde{\beta}$ regarded as an element of $A\otimes A$ is a solution of the EAYBE of mass $\frac{\kappa+1}{4}$ if and only if α is an extended $\mathcal{O}$-operator with modification β of mass k.*
  (b) *If $\kappa = -1$, then $r_\pm = \tilde{\alpha}\pm\tilde{\beta}$ regarded as an element of $A\otimes A$ is a solution of the AYBE if and only if α is an extended $\mathcal{O}$-operator with modification β of mass -1.*
  (c) *If $\kappa = 0$, then $\tilde{\alpha}$ regarded as an element of $A\otimes A$ is a solution of the AYBE if and only if α is a Rota-Baxter operator of weight zero.*

Proof. (i) Since B is symmetric and invariant, for any $x, y, z \in A$, we have

$$B(x\cdot y, z) = B(x, y\cdot z) \Leftrightarrow \langle\varphi(x\cdot y), z\rangle = \langle\varphi(x), y\cdot z\rangle \Leftrightarrow \varphi(xR(y)) = \varphi(x)L^*(y), \tag{54}$$

$$B(z, x\cdot y) = B(y\cdot z, x) \Leftrightarrow \langle\varphi(z), x\cdot y\rangle = \langle\varphi(y\cdot z), x\rangle \Leftrightarrow R^*(y)\varphi(z) = \varphi(L(y)z). \tag{55}$$

On the other hand, since φ is invertible, for any $a^*, b^* \in A^*$, there exist $x, y \in A$ such that $\varphi(x) = a^*, \varphi(y) = b^*$. So according to Eq. (54) and Eq. (55), the equation

$$\tilde{\alpha}(a^*)\cdot\tilde{\alpha}(b^*) - \tilde{\alpha}(\varphi(\tilde{\alpha}(a^*)\cdot\varphi^{-1}(b^*) + \varphi^{-1}(a^*)\cdot\tilde{\alpha}(b^*))) = \kappa\tilde{\beta}(a^*)\cdot\tilde{\beta}(b^*),$$

is equivalent to

$$\tilde{\alpha}(a^*)\cdot\tilde{\alpha}(b^*) - \tilde{\alpha}(R^*(\tilde{\alpha}(a^*))b^* + a^*L^*(\tilde{\alpha}(b^*))) = \kappa\tilde{\beta}(a^*)\cdot\tilde{\beta}(b^*).$$

By Lemma 3.10, $\beta : A \to A$ is a balanced A-bimodule homomorphism if and only if $\tilde{\beta} : A^* \to A$ is a balanced A-bimodule homomorphism. So α is an extended $\mathcal{O}$-operator with modification β of mass κ if and only if $\tilde{\alpha}$ is an extended $\mathcal{O}$-operator with modification $\tilde{\beta}$ of mass κ.

(ii) If α is skew-adjoint with respect to B, then

$$\langle\alpha(x), \varphi(y)\rangle + \langle\varphi(x), \alpha(y)\rangle = 0, \quad \forall x, y \in A.$$

Hence $\langle\tilde{\alpha}(a^*), b^*\rangle + \langle a^*, \tilde{\alpha}(b^*)\rangle = 0$ for any $a^*, b^* \in A^*$. So $\tilde{\alpha}$ regarded as an element of $A\otimes A$ is skew-symmetric.

Then by Theorem 3.5, Item (iia) holds. By Corollary 3.6 Item (iib) and Item (iic) hold. □

Corollary 3.13. *Let $\mathbf{k}$ be a field of characteristic not equal to 2. Let A be a $\mathbf{k}$-algebra of finite $\mathbf{k}$-dimension and let $r \in A\otimes A$. Define $\alpha, \beta \in A\otimes A$ by Eq. (43). Then $r = \alpha + \beta$. Let $B : A\otimes A \to \boldsymbol{k}$ be a nondegenerate symmetric and invariant bilinear form. Define the linear map $\varphi : A \to A^*$ by Eq. (53).*

(i) *Suppose that $\beta \in A\otimes A$ is invariant. Then r is a solution of the EAYBE of mass $\frac{\kappa+1}{4}$ if and only if $\hat{\alpha} = \alpha\varphi : A \to A$ is an extended $\mathcal{O}$-operator with modification $\hat{\beta} = \beta\varphi : A \to A$ of mass k.*

(ii) *Suppose that $\beta \in A\otimes A$ is invariant. Then r is a solution of the AYBE if and only if $\hat{\alpha} = \alpha\varphi : A \to A$ is an extended $\mathcal{O}$-operator with modification $\hat{\beta} = \beta\varphi : A \to A$ of mass -1. In particularly, if in addition, $\beta = 0$, i.e., r is skew-symmetric, then r is a solution of the AYBE if and only if $\hat{\alpha} = \hat{r} = r\varphi : A \to A$ is a Rota-Baxter operator of weight zero.*

Proof. Since $\alpha \in A\otimes A$ is skew-symmetric, for any $x, y \in A$,

$$\begin{aligned} & \langle \alpha, \varphi(x)\otimes\varphi(y)\rangle = -\langle \alpha, \varphi(y)\otimes\varphi(x)\rangle \\ \Leftrightarrow\quad & \langle \alpha(\varphi(x)), \varphi(y)\rangle = -\langle \alpha(\varphi(y)), \varphi(x)\rangle \\ \Leftrightarrow\quad & B(\hat{\alpha}(x), y) = -B(\hat{\alpha}(y), x). \end{aligned}$$

Thus $\hat{\alpha} = \alpha\varphi$ is skew-adjoint with respect to $B(\ ,\ )$. On the other hand, from the proof of Corollary 3.11, we show that $\hat{\beta} = \beta\varphi$ is self-adjoint with respect to $B(\ ,\ )$. So the conclusion follows from Theorem 3.12. $\square$

3.4. **Extended $\mathcal{O}$-operators in general and EAYBE.** We now establish the relationship between an extended $\mathcal{O}$-operator $\alpha : V \to A$ in general and the AYBE. For this purpose we prove that an extended $\mathcal{O}$-operator $\alpha : V \to A$ naturally gives rise to an extended $\mathcal{O}$-operator on a larger associative algebra $\hat{A}$ associated to the dual bimodule $(\hat{A}^*, R^*_{\hat{A}}, L^*_{\hat{A}})$. We first introduce some notations.

Definition 3.14. *Let A be a $\mathbf{k}$-algebra and let (V, ℓ, r) be an A-bimodule, both with finite $\mathbf{k}$-dimension. Let (V^*, r^*, ℓ^*) be the dual A-bimodule and let $\hat{A} = A \ltimes_{r^*,\ell^*} V^*$. Identify a linear map $\gamma : V \to A$ as an element in $\hat{A}\otimes\hat{A}$ through the injective map*

$$\mathrm{Hom}_{\mathbf{k}}(V, A) \cong A\otimes V^* \hookrightarrow \hat{A}\otimes\hat{A}. \tag{56}$$

Denote

$$\tilde{\gamma}_{\pm} := \gamma \pm \gamma^{21}, \tag{57}$$

where $\gamma^{21} = \sigma(\gamma) \in V^*\otimes A \subset \hat{A}\otimes\hat{A}$ with $\sigma : A\otimes V^* \to V^*\otimes A, a\otimes u^* \to u^*\otimes a$, being the switch operator.

Lemma 3.15. *Let A be a $\mathbf{k}$-algebra and let (V, ℓ, r) be an A-bimodule, both with finite $\mathbf{k}$-dimension. Suppose that $\beta : V \to A$ is a linear map which is identified as an element in $\hat{A}\otimes\hat{A}$ by Eq. (56). Define $\tilde{\beta}_+$ by Eq. (57). Then $\tilde{\beta}_+$, identified as a linear map from $\hat{A}^*$ to $\hat{A}$, is a balanced $\hat{A}$-bimodule homomorphism from $(\hat{A}^*, R^*_{\hat{A}}, L^*_{\hat{A}})$ to $(\hat{A}, L_{\hat{A}}, R_{\hat{A}})$ if and only if $\beta : V \to A$ is a balanced A-bimodule homomorphism from (V, ℓ, r) to (A, L_A, R_A).*

Proof. For the linear map $\tilde{\beta}_+ : \hat{A}^* \to \hat{A}$, we have $\tilde{\beta}_+(a^*) = \beta^*(a^*)$ for $a^* \in A^*$ and $\tilde{\beta}_+(u) = \beta(u)$ for $u \in V$, where $\beta^* : A^* \to V^*$ is the dual linear map associated to β given by

$$\langle \beta^*(a^*), v\rangle = \langle a^*, \beta(v)\rangle,\quad \forall a^* \in A^*, v \in V.$$

First suppose that $\beta : (V, \ell, r) \to A$ is a balanced A-bimodule homomorphism. Let $b^* \in A^*, v \in V$, then

$$R^*_{\hat{A}}(\tilde{\beta}_+(a^*+u))(b^*+v) = R^*_{\hat{A}}(\beta^*(a^*))b^* + R^*_{\hat{A}}(\beta^*(a^*))v + R^*_{\hat{A}}(\beta(u))b^* + R^*_{\hat{A}}(\beta(u))v,$$

$$(a^*+u)L^*_{\hat{A}}(\tilde{\beta}_+(b^*+v)) = a^*L^*_{\hat{A}}(\beta^*(b^*)) + a^*L^*_{\hat{A}}(\beta(v)) + uL^*_{\hat{A}}(\beta^*(b^*)) + uL^*_{\hat{A}}(\beta(v)).$$

On the other hand, for any $x \in A$, $w^* \in V^*$,

$$\begin{aligned}
&\langle R^*_{\hat{A}}(\beta^*(a^*))b^* - a^*L^*_{\hat{A}}(\beta^*(b^*)), x\rangle = \langle b^*, x\cdot\beta^*(a^*)\rangle - \langle a^*, \beta^*(b^*)\cdot x\rangle = 0,\\
&\langle R^*_{\hat{A}}(\beta^*(a^*))b^* - a^*L^*_{\hat{A}}(\beta^*(b^*)), w^*\rangle = \langle b^*, w^*\cdot\beta^*(a^*)\rangle - \langle a^*, \beta^*(b^*)\cdot w^*\rangle = 0,\\
&\langle R^*_{\hat{A}}(\beta^*(a^*))v - a^*L^*_{\hat{A}}(\beta(v)), x\rangle = \langle v, x\cdot\beta^*(a^*)\rangle - \langle a^*, \beta(v)\cdot x\rangle = \langle a^*, \beta(vr(x)) - \beta(v)\cdot x\rangle = 0,\\
&\langle R^*_{\hat{A}}(\beta^*(a^*))v - a^*L^*_{\hat{A}}(\beta(v)), w^*\rangle = \langle v, w^*\cdot\beta^*(a^*)\rangle - \langle a^*, \beta(v)\cdot w^*\rangle = 0,\\
&\langle R^*_{\hat{A}}(\beta(u))b^* - uL^*_{\hat{A}}(\beta^*(b^*)), x\rangle = \langle b^*, x\cdot\beta(u)\rangle - \langle u, \beta^*(b^*)\cdot x\rangle = \langle b^*, x\cdot\beta(u) - \beta(l(x)u)\rangle = 0,\\
&\langle R^*_{\hat{A}}(\beta(u))b^* - uL^*_{\hat{A}}(\beta^*(b^*)), w^*\rangle = \langle b^*, w^*\cdot\beta(u)\rangle - \langle u, \beta^*(b^*)\cdot w^*\rangle = 0.\\
&\langle R^*_{\hat{A}}(\beta(u))v - uL^*_{\hat{A}}(\beta(v)), w^*\rangle = \langle v, w^*\cdot\beta(u)\rangle - \langle u, \beta(v)\cdot w^*\rangle = \langle \ell(\beta(u))v - ur(\beta(v)), w^*\rangle = 0,\\
&\langle R^*_{\hat{A}}(\beta(u))v - uL^*_{\hat{A}}(\beta(v)), x\rangle = \langle v, x\cdot\beta(u)\rangle - \langle u, \beta(v)\cdot x\rangle = 0.
\end{aligned}$$

Therefore, $R^*_{\hat{A}}(\tilde{\beta}_+(a^*+u))(b^*+v) = (a^*+u)L^*_{\hat{A}}(\tilde{\beta}_+(b^*+v))$. Since $\tilde{\beta}_+ \in \hat{A}\otimes\hat{A}$ is symmetric, by Lemma 3.2, $\tilde{\beta}_+$, identified as a linear map from $\hat{A}^*$ to $\hat{A}$, is a balanced $\hat{A}$-bimodule homomorphism from $(\hat{A}^*, R^*_{\hat{A}}, L^*_{\hat{A}})$ to $(\hat{A}, L_{\hat{A}}, R_{\hat{A}})$.

Conversely, if $\tilde{\beta}_+$, identified as a linear map from $\hat{A}^*$ to $\hat{A}$, is a balanced $\hat{A}$-bimodule homomorphism from $(\hat{A}^*, R^*_{\hat{A}}, L^*_{\hat{A}})$ to $(\hat{A}, L_{\hat{A}}, R_{\hat{A}})$, then for any $u, v \in V, x \in A$,

$$\begin{aligned}
R^*_{\hat{A}}(\tilde{\beta}_+(u))v = uL^*_{\hat{A}}(\tilde{\beta}_+(v)) \quad&\Leftrightarrow\quad \ell(\beta(u))v = ur(\beta(v)),\\
\tilde{\beta}_+(R^*_{\hat{A}}(x)v) = x\cdot\tilde{\beta}_+(v) \quad&\Leftrightarrow\quad \beta(\ell(x)v) = x\cdot\beta(v),\\
\tilde{\beta}_+(uL^*_{\hat{A}}(x)) = \tilde{\beta}_+(u)\cdot x \quad&\Leftrightarrow\quad \beta(ur(x)) = \beta(u)\cdot x.
\end{aligned}$$

So $\beta : (V, \ell, r) \to (A, L_A, R_A)$ is a balanced A-bimodule homomorphism. $\square$

Theorem 3.16. *Let A be a* $\mathbf{k}$*-algebra and let (V, ℓ, r) be an A-bimodule, both with finite* $\mathbf{k}$*-dimension. Let $\alpha, \beta : V \to A$ be two* $\mathbf{k}$*-linear maps. Let $\tilde{\alpha}_-$ and $\tilde{\beta}_+$ be defined by Eq. (56) and identified as linear maps from $\hat{A}^*$ to $\hat{A}$. Then α is an extended $\mathcal{O}$-operator with modification β of mass κ if and only if $\tilde{\alpha}_-$ is an extended $\mathcal{O}$-operator with modification $\tilde{\beta}_+$ of mass κ.*

Proof. Note that for any $a^* \in A^*, v \in V$, $\tilde{\alpha}_-(a^*) = \alpha^*(a^*)$ and $\tilde{\alpha}_-(v) = -\alpha(v)$, where $\alpha^* : A^* \to V^*$ is the dual linear map of α. Suppose that α is an extended $\mathcal{O}$-operator with modification β of mass κ. Then for any $a^*, b^* \in A^*$, $u, v \in V$, we have

$$\begin{aligned}
&\tilde{\alpha}_-(u+a^*)\cdot\tilde{\alpha}_-(v+b^*) - \tilde{\alpha}_-(R^*_{\hat{A}}(\tilde{\alpha}_-(u+a^*))(v+b^*) + (u+a^*)L^*_{\hat{A}}(\tilde{\alpha}_-(v+b^*)))\\
= \;& \alpha(u)\cdot\alpha(v) - \alpha(u)\cdot\alpha^*(b^*) - \alpha^*(a^*)\cdot\alpha(v) + \alpha^*(a^*)\cdot\alpha^*(b^*)\\
&-\tilde{\alpha}_-(-R^*_{\hat{A}}(\alpha(u))v - R^*_{\hat{A}}(\alpha(u))b^* + R^*_{\hat{A}}(\alpha^*(a^*))v + R^*_{\hat{A}}(\alpha^*(a^*))b^* - uL^*_{\hat{A}}(\alpha(v))\\
&-a^*L^*_{\hat{A}}(\alpha(v)) + uL^*_{\hat{A}}(\alpha^*(b^*)) + a^*L^*_{\hat{A}}(\alpha^*(b^*)))\\
= \;& \alpha(u)\cdot\alpha(v) - \alpha(\ell(\alpha(u))v) - \alpha(ur(\alpha(v))) - r^*(\alpha(u))\alpha^*(b^*) + \alpha^*(R^*_{\hat{A}}(\alpha(u))b^*)\\
&-\alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*))) - \alpha^*(a^*)\ell^*(\alpha(v)) - \alpha^*(R^*_{\hat{A}}(\alpha^*(a^*))v) + \alpha^*(a^*L^*_{\hat{A}}(\alpha(v))).
\end{aligned}$$

On the other hand, for any $w \in V$ we have

$$\begin{aligned}
&\langle -r^*(\alpha(u))\alpha^*(b^*) + \alpha^*(R^*_{\hat{A}}(\alpha(u))b^*) - \alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*))), w\rangle\\
= \;& \langle b^*, \alpha(w)\cdot\alpha(u) - \alpha(\ell(\alpha(w))u + wr(\alpha(u))\rangle = \langle b^*, \kappa\beta(w)\cdot\beta(u)\rangle\\
= \;& \langle b^*, \kappa\beta(wr(\beta(u)))\rangle = \langle \kappa r^*(\beta(u))\beta^*(b^*), w\rangle.
\end{aligned}$$

Therefore

$$-r^*(\alpha(u))\alpha^*(b^*) + \alpha^*(R^*_{\hat{A}}(\alpha(u))b^*) - \alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*))) = \kappa r^*(\beta(u))\beta^*(b^*).$$

Similarly,

$$\begin{aligned} &\langle -\alpha^*(a^*)\ell^*(\alpha(v)) - \alpha^*(R^*_{\hat{A}}(\alpha^*(a^*))v) + \alpha^*(a^*L^*_{\hat{A}}(\alpha(v))), w\rangle \\ = \ &\langle a^*, \alpha(v)\cdot\alpha(w) - \alpha(\ell(\alpha(v))w + vr(\alpha(w)))\rangle = \langle a^*, \kappa\beta(v)\cdot\beta(w)\rangle \\ = \ &\langle a^*, \kappa\beta(\ell(\beta(v))w)\rangle = \langle \kappa\beta^*(a^*)\ell^*(\beta(v)), w\rangle. \end{aligned}$$

Hence

$$-\alpha^*(a^*)\ell^*(\alpha(v)) - \alpha^*(R^*_{\hat{A}}(\alpha^*(a^*))v) + \alpha^*(a^*L^*_{\hat{A}}(\alpha(v))) = \kappa\beta^*(a^*)\ell^*(\beta(v)).$$

So

$$\begin{aligned} &\tilde{\alpha}_-(u+a^*)\cdot\tilde{\alpha}_-(v+b^*) - \tilde{\alpha}_-(R^*_{\hat{A}}(\tilde{\alpha}_-(u+a^*))(v+b^*) + (u+a^*)L^*_{\hat{A}}(\tilde{\alpha}_-(v+b^*))) \\ = \ &\kappa\beta(u)\cdot\beta(v) + \kappa r^*(\beta(u))\beta^*(b^*) + \kappa\beta^*(a^*)\ell^*(\beta(v)) \\ = \ &\kappa\beta(u)\cdot\beta(v) + \kappa\beta(u)\cdot\beta^*(b^*) + \kappa\beta^*(a^*)\cdot\beta(v) = \kappa\tilde{\beta}_+(u+a^*)\tilde{\beta}_+(v+b^*). \end{aligned}$$

If $\kappa = 0$, then the above equation implies that $\tilde{\alpha}_-$ is an $\mathcal{O}$-operator of weight zero. If $\kappa \neq 0$, then β is a balanced A-bimodule homomorphism, which, according to Lemma 3.15, implies that $\tilde{\beta}_+$ from $(\hat{A}^*, R^*_{\hat{A}}, L^*_{\hat{A}})$ to $\hat{A}$ is a balanced $\hat{A}$-bimodule homomorphism. So $\tilde{\alpha}_-$ is an extended $\mathcal{O}$-operator with modification $\tilde{\beta}_+$ of mass κ.

Conversely, if $\tilde{\alpha}_-$ is an extended $\mathcal{O}$-operator with modification $\tilde{\beta}_+$ of mass κ. If $\kappa \neq 0$, then $\tilde{\beta}_+$ from $(\hat{A}^*, R^*_{\hat{A}}, L^*_{\hat{A}})$ to $\hat{A}$ is a balanced $\hat{A}$-bimodule homomorphism, which by Lemma 3.15 implies that β from (V, ℓ, r) to A is a balanced A-bimodule homomorphism. Moreover, for any $u, v \in V$ we have

$$\tilde{\alpha}_-(u)\cdot\tilde{\alpha}_-(v) - \tilde{\alpha}_-(R^*_{\hat{A}}(\tilde{\alpha}_-(u))v + uL^*_{\hat{A}}(\tilde{\alpha}_-(v))) = \kappa\tilde{\beta}_+(u)\tilde{\beta}_+(v), \tag{58}$$

which implies that α is an extended $\mathcal{O}$-operator with modification β of mass κ. If $\kappa = 0$, then Eq. (58) for $\kappa = 0$ implies that α is an $\mathcal{O}$-operator of weight zero. So the conclusion follows. $\square$

By Theorem 3.16, the results from the previous sections on $\mathcal{O}$-operators on A can be applied to general $\mathcal{O}$-operators.

Corollary 3.17. *Let A be a $\mathbf{k}$-algebra and let V be an A-bimodule, both with finite $\mathbf{k}$-dimension.*

(i) *Suppose the characteristic of the field $\mathbf{k}$ is not 2. Let $\alpha, \beta : V \to A$ be linear maps which are identified as elements in $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$. Then α is an extended $\mathcal{O}$-operator with modification β of mass k if and only if $(\alpha - \alpha^{21}) \pm (\beta + \beta^{21})$ is a solution of the EAYBE of mass $\frac{\kappa+1}{4}$ in $A \ltimes_{r^*,\ell^*} V^*$.*

(ii) *Let $\alpha : V \to A$ be a linear map which is identified as an element in $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$. Then α is an $\mathcal{O}$-operator of weight zero if and only if $\alpha - \alpha^{21}$ is a skew-symmetric solution of the AYBE in Eq. (40) in $A \ltimes_{r^*,\ell^*} V^*$. In particular, a linear map $P : A \to A$ is a Rota-Baxter operator of weight zero if and only if $r = P - P^{21}$ is a skew-symmetric solution of the AYBE in $A \ltimes_{R^*,L^*} A^*$.*

(iii) *Let $\alpha, \beta : V \to A$ be two linear maps which are identified as elements in $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$. Then α is an extended $\mathcal{O}$-operator with modification β of mass -1 if and only if $(\alpha - \alpha^{21}) \pm (\beta + \beta^{21})$ is a solution of the AYBE in $A \ltimes_{r^*,\ell^*} V^*$.*

(iv) *Let $\alpha : A \to A$ be a linear map which is identified as an element in $(A \ltimes_{R^*,L^*} A^*) \otimes (A \ltimes_{R^*,L^*} A^*)$. Then α satisfies Eq. (36) if and only if $(\alpha - \alpha^{21}) \pm (\mathrm{id} + \mathrm{id}^{21})$ is a solution of the AYBE in $A \ltimes_{R^*,L^*} A^*$.*

(v) *Let $P : A \to A$ be a linear map which is identified as an element of $A \ltimes_{R^*,L^*} A^*$. Then P is a Rota-Baxter operator of weight $\lambda \neq 0$ if and only if both $\frac{2}{\lambda}(P - P^{21}) + 2\mathrm{id}$ and $\frac{2}{\lambda}(P - P^{21}) - 2\mathrm{id}^{21}$ are solutions of the AYBE in $A \ltimes_{R^*,L^*} A^*$.*

Proof. (i) This follows from Theorem 3.16 and Theorem 3.5.

(ii) This follows from Theorem 3.16 for $\kappa = 0$ (or $\beta = 0$) and Corollary 3.6.

(iii) This follows from Theorem 3.16 for $\kappa = -1$ and Corollary 3.6.

(iv) This follows from Item (iii) in the case that $(V, r, \ell) = (A, L, R)$ and $\beta = \mathrm{id}$.

(v) By [11] (see the discussion after Corollary 2.18), P is a Rota-Baxter operator of weight $\lambda \neq 0$ if and only if $\frac{2P}{\lambda} + \mathrm{id}$ is an extended $\mathcal{O}$-operator with modification id of mass -1 from (A, L, R) to A, i.e., $\frac{2P}{\lambda} + \mathrm{id}$ satisfies Eq. (36). Then the conclusion follows from Item (iv). $\square$

4. Extended $\mathcal{O}$-operators and the generalized associative Yang-Baxter equation

We define the generalized associative Yang-Baxter equation and study its relationship with extended $\mathcal{O}$-operators.

4.1. **Generalized associative Yang-Baxter equation.** We adapt the same notations as in Definition 3.1.

Definition 4.1. *An element $r \in A \otimes A$ is called a solution of the* **generalized associative Yang-Baxter equation (GAYBE) in** A *if it satisfies the relation*

$$(\mathrm{id} \otimes \mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id} \otimes \mathrm{id})(r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}) = 0, \quad \forall x \in A. \tag{59}$$

Our definition is motivated by the following fact (also see Proposition 5.1 in [1]) which relates to the construction of a kind of bialgebras in the different notions such that associative D-bialgebras [29], balanced infinitesimal bialgebras (in the opposite algebras) [3] and antisymmetric infinitesimal bialgebras [5].

Proposition 4.2. ([1, 3, 5]) *Let A be a $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension and let $r \in A \otimes A$. Define $\Delta : A \to A \otimes A$ by*

$$\Delta(x) = (\mathrm{id} \otimes L(x) - R(x) \otimes \mathrm{id})r, \quad \forall x \in A. \tag{60}$$

Then

$$\Delta^* : A^* \otimes A^* \hookrightarrow (A \otimes A)^* \to A^* \tag{61}$$

defines an associative multiplication on A^* if and only if r is a solution of the GAYBE.

Moreover, we have the following conclusion

Lemma 4.3. *Let $(A, \cdot)$ be a $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension. Let $r \in A \otimes A$. The multiplication $*$ on A^* defined by Eq. (61) is also given by*

$$a^* * b^* = R^*(r(a^*))b^* - L^*(r^t(b^*))a^*, \quad \forall a^*, b^* \in A^*. \tag{62}$$

Proof. Let $\{e_1, ..., e_n\}$ be a basis of A and $\{e_1^*, ..., e_n^*\}$ be its dual basis. Suppose that $r = \sum_{i,j} a_{i,j} e_i \otimes e_j$ and $e_i \cdot e_j = \sum_k c_{i,j}^k e_k$. Then for any k, l we have

$$\begin{aligned} e_k^* * e_l^* &= \sum_s \langle e_k^* \otimes e_l^*, \Delta(e_s)\rangle e_s^* = \sum_s \langle e_k^* \otimes e_l^*, (\mathrm{id} \otimes L(e_s) - R(e_s) \otimes \mathrm{id}) r\rangle e_s^* \\ &= \sum_{s,t} (a_{k,t} c_{s,t}^l - c_{t,s}^k a_{t,l}) e_s^* = R^*(r(e_k^*)) e_l^* - L^*(r^t(e_l^*)) e_k^*. \end{aligned}$$

□

The above lemma motivates us to apply the approach considered in Section 2.2. More precisely, we take the A-bimodule $\mathbf{k}$-algebra $(R, \circ, \ell, r)$ to be (A^*, R^*, L^*) with the zero multiplication and set

$$\delta_+ = r, \quad \delta_- = -r^t. \tag{63}$$

Assume that $\mathbf{k}$ has characteristic not equal to 2 and define

$$\alpha = (r - r^t)/2, \quad \beta = (r + r^t)/2, \tag{64}$$

that is, α and β are the **skew-symmetric part** and the **symmetric part** of r respectively. So $r = \alpha + \beta$ and $r^t = -\alpha + \beta$.

Proposition 4.4. *Let $\mathbf{k}$ have characteristic not equal to 2. Let $(A, \cdot)$ be a $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension and $r \in A \otimes A$. Let α, β be given by Eq. (64). Suppose that β is a balanced A-bimodule homomorphism, that is, β satisfies Eq. (44). If α is an extended $\mathcal{O}$-operator with modification β of any mass $\kappa \in \mathbf{k}$, then the product defined by Eq. (62) defines a $\mathbf{k}$-algebra structure on A^* and r is a solution of the GAYBE.*

Proof. Applying Theorem 2.14 to the A-bimodule $\mathbf{k}$-algebra $(R, \circ, \ell, r)$ constructed before the proposition, we see that the product defined by Eq. (62) is associative. Then by Lemma 4.3, r is a solution of the GAYBE. □

As a direct consequence, we have

Corollary 4.5. *Under the same assumption as in Proposition 4.4, a solution of the EAYBE of any mass $\kappa \in \mathbf{k}$ is also a solution of the GAYBE.*

Proof. Let r be a solution of the EAYBE of mass κ. Define α and β by Eq. (64). Then by Theorem 3.5, α is an extended $\mathcal{O}$-operator with modification β of mass $4\kappa - 1$. Then by Proposition 4.4, r is a solution of the GAYBE. □

4.2. **Relation with extended $\mathcal{O}$-operators.**

Lemma 4.6. *Let A be a $\mathbf{k}$-algebra and (V, ℓ, r) be a bimodule. Let $\alpha : V \to A$ be a linear map. Then the product*

$$u *_\alpha v := \ell(\alpha(u))v + ur(\alpha(v)), \quad \forall u, v \in V, \tag{65}$$

defines a $\mathbf{k}$*-algebra structure on* V *if and only if the following equation holds:*

$$\ell\Big(\alpha(u)\cdot\alpha(v)-\alpha(u*_\alpha v)\Big)w = ur\Big(\alpha(v)\cdot\alpha(w)-\alpha(v*_\alpha w)\Big),\quad \forall u,v\in V. \tag{66}$$

Proof. It follows from Lemma 2.12 by setting $(R,\ell,r)=(V,\ell,r)$ and $\lambda=0$. □

Theorem 4.7. *Let* A *be a* $\mathbf{k}$*-algebra and* (V,ℓ,r) *be an* A*-bimodule, both of finite dimension over* $\mathbf{k}$*. Let* $\alpha: V\to A$ *be a linear map from* V *to* A*. Using the same notations in Definition 3.14, then* $\tilde{\alpha}_-$ *identified as an element of* $\hat{A}\otimes\hat{A}$ *is a skew-symmetric solution of the GAYBE (59) if and only if Eq. (66) and the following equations hold:*

$$\alpha(u)\cdot\alpha(\ell(x)v)-\alpha(u*_\alpha(\ell(x)v)) = \alpha(ur(x))\cdot\alpha(v)-\alpha((ur(x))*_\alpha v), \tag{67}$$

$$\alpha(u)\cdot\alpha(vr(x))-\alpha(u*_\alpha(vr(x))) = (\alpha(u)\cdot\alpha(v))\cdot x-\alpha(u*_\alpha v)\cdot x, \tag{68}$$

$$\alpha(\ell(x)u)\cdot\alpha(v)-\alpha((\ell(x)u)*_\alpha v) = x\cdot(\alpha(u)\cdot\alpha(v))-x\cdot\alpha(u*_\alpha v), \tag{69}$$

for any $u,v\in V, x\in A$.

Proof. By Proposition 4.2, Lemma 4.3 and Lemma 4.6 we see that $\tilde{\alpha}_-\in\hat{A}\otimes\hat{A}$ is a skew-symmetric solution of the GAYBE (59) if and only if for any $u,v,w\in V, a^*,b^*,c^*\in A^*$,

$$\begin{aligned}
& R^*_{\hat{A}}(\tilde{\alpha}_-(u+a^*)\cdot\tilde{\alpha}_-(v+b^*)-\tilde{\alpha}_-(R^*_{\hat{A}}(\tilde{\alpha}_-(u+a^*))(v+b^*)+(u+a^*)L^*_{\hat{A}}(\tilde{\alpha}_-(v+b^*))))(w+c^*)\\
= \;& (u+a^*)L^*_{\hat{A}}(\tilde{\alpha}_-(v+b^*)\cdot\tilde{\alpha}_-(w+c^*)-\tilde{\alpha}_-(R^*_{\hat{A}}(\tilde{\alpha}_-(v+b^*))(w+c^*)+(v+b^*)L^*_{\hat{A}}(\tilde{\alpha}_-(w+c^*)))),
\end{aligned}$$

By the proof of Theorem 3.16, the above equation is equivalent to

$$\begin{aligned}
& R^*_{\hat{A}}(\alpha(u)\cdot\alpha(v)-\alpha(\ell(\alpha(u))v)-\alpha(ur(\alpha(v)))-r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\hat{A}}(\alpha(u))b^*)\\
& -\alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*)))-\alpha^*(a^*)\ell^*(\alpha(v))-\alpha^*(R^*_{\hat{A}}(\alpha^*(a^*))v)+\alpha^*(a^*L^*_{\hat{A}}(\alpha(v))))w+\\
& R^*_{\hat{A}}(\alpha(u)\cdot\alpha(v)-\alpha(\ell(\alpha(u))v)-\alpha(ur(\alpha(v)))-r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\hat{A}}(\alpha(u))b^*)\\
& -\alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*)))-\alpha^*(a^*)\ell^*(\alpha(v))-\alpha^*(R^*_{\hat{A}}(\alpha^*(a^*))v)+\alpha^*(a^*L^*_{\hat{A}}(\alpha(v))))c^*\\
= \;& uL^*_{\hat{A}}(\alpha(v)\cdot\alpha(w)-\alpha(\ell(\alpha(v))w)-\alpha(vr(\alpha(w)))-r^*(\alpha(v))\alpha^*(c^*)+\alpha^*(R^*_{\hat{A}}(\alpha(v))c^*)\\
& -\alpha^*(vL^*_{\hat{A}}(\alpha^*(c^*)))-\alpha^*(b^*)\ell^*(\alpha(w))-\alpha^*(R^*_{\hat{A}}(\alpha^*(b^*))w)+\alpha^*(b^*L^*_{\hat{A}}(\alpha(w))))+\\
& a^*L^*_{\hat{A}}(\alpha(v)\cdot\alpha(w)-\alpha(\ell(\alpha(v))w)-\alpha(vr(\alpha(w)))-r^*(\alpha(v))\alpha^*(c^*)+\alpha^*(R^*_{\hat{A}}(\alpha(v))c^*)\\
& -\alpha^*(vL^*_{\hat{A}}(\alpha^*(c^*)))-\alpha^*(b^*)\ell^*(\alpha(w))-\alpha^*(R^*_{\hat{A}}(\alpha^*(b^*))w)+\alpha^*(b^*L^*_{\hat{A}}(\alpha(w)))).
\end{aligned}$$

By suitable choices of $u,v,w\in V$ and $a^*,b^*,c^*\in A^*$, we find that this equation holds if and only if the following equations hold:

$$\begin{aligned}
& R^*_{\hat{A}}(\alpha(u)\cdot\alpha(v)-\alpha(\ell(\alpha(u))v+ur(\alpha(v))))w\\
= \;& uL^*_{\hat{A}}(\alpha(v)\cdot\alpha(w)-\alpha(\ell(\alpha(v))w+vr(\alpha(w))))\quad (\text{take } a^*=b^*=c^*=0),
\end{aligned} \tag{70}$$

$$\begin{aligned}
& R^*_{\hat{A}}(-r^*(\alpha(u))\alpha^*(b^*)+\alpha^*(R^*_{\hat{A}}(\alpha(u))b^*)-\alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*))))w\\
= \;& uL^*_{\hat{A}}(-\alpha^*(b^*)\ell^*(\alpha(w))-\alpha^*(R^*_{\hat{A}}(\alpha^*(b^*))w)+\alpha^*(b^*L^*_{\hat{A}}(\alpha(w))))\quad (\text{take } v=a^*=c^*=0),
\end{aligned} \tag{71}$$

$$\begin{aligned}
& R^*_{\hat{A}}(-\alpha^*(a^*)\ell^*(\alpha(v))-\alpha^*(R^*_{\hat{A}}(\alpha^*(a^*))v)+\alpha^*(a^*L^*_{\hat{A}}(\alpha(v))))w\\
= \;& a^*L^*_{\hat{A}}(\alpha(v)\cdot\alpha(w)-\alpha(\ell(\alpha(v))w)-\alpha(vr(\alpha(w))))\quad (\text{take } u=b^*=c^*=0),
\end{aligned} \tag{72}$$

$$\begin{aligned}
& R^*_{\hat{A}}(\alpha(u)\cdot\alpha(v)-\alpha(\ell(\alpha(u))v)-\alpha(ur(\alpha(v))))c^*\\
= \;& uL^*_{\hat{A}}(-r^*(\alpha(v))\alpha^*(c^*)+\alpha^*(R^*_{\hat{A}}(\alpha(v))c^*)-\alpha^*(vL^*_{\hat{A}}(\alpha^*(c^*))))\quad (\text{take } w=a^*=b^*=0),
\end{aligned} \tag{73}$$

$$\begin{aligned} & R^*_{\hat{A}}(-r^*(\alpha(u))\alpha^*(b^*) + \alpha^*(R^*_{\hat{A}}(\alpha(u))b^*) - \alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*))))c^* \\ = & \ 0 \quad (\text{take } v = w = a^* = 0), \end{aligned} \tag{74}$$

$$\begin{aligned} & R^*_{\hat{A}}(-\alpha^*(a^*)\ell^*(\alpha(v)) - \alpha^*(R^*_{\hat{A}}(\alpha^*(a^*))v) + \alpha^*(a^*L^*_{\hat{A}}(\alpha(v))))c^* \\ = & \ a^*L^*_{\hat{A}}(-r^*(\alpha(v))\alpha^*(c^*) + \alpha^*(R^*_{\hat{A}}(\alpha(v))c^*) - \alpha^*(vL^*_{\hat{A}}(\alpha^*(c^*)))) \quad (\text{take } u = w = b^* = 0), \end{aligned} \tag{75}$$

$$\begin{aligned} & a^*L^*_{\hat{A}}(-\alpha^*(b^*)\ell^*(\alpha(w)) - \alpha^*(R^*_{\hat{A}}(\alpha^*(b^*))w) + \alpha^*(b^*L^*_{\hat{A}}(\alpha(w)))) \\ = & \ 0 \quad (\text{take } u = v = c^* = 0). \end{aligned} \tag{76}$$

Thus we just need to prove

(i) Eq. (70) $\Leftrightarrow$ Eq. (66),
(ii) Eq. (71) $\Leftrightarrow$ Eq. (67),
(iii) Eq. (72) $\Leftrightarrow$ Eq. (68),
(iv) Eq. (73) $\Leftrightarrow$ Eq. (69),
(v) both sides of Eq. (75) equal to zero, and
(vi) Eq. (74) and Eq. (76) hold.

The proofs of these statements are similar. So we just prove that Eq. (71) holds if and only if Eq. (67) holds. Let LHS and RHS denote the left-hand side and right-hand side of Eq. (71) respectively. Then for any $x \in A, s^* \in V^*$, we have

$$\langle LHS, s^*\rangle = \langle RHS, s^*\rangle = 0.$$

Further

$$\begin{aligned} \langle LHS, x\rangle &= \langle w, -r^*(x)(r^*(\alpha(u))\alpha^*(b^*)) + r^*(x)\alpha^*(R^*_{\hat{A}}(\alpha(u))b^*) - r^*(x)\alpha^*(uL^*_{\hat{A}}(\alpha^*(b^*)))\rangle \\ &= \langle -\alpha((wr(x))r(\alpha(u))) + \alpha(wr(x))\cdot\alpha(u), b^*\rangle - \langle \alpha^*(b^*)\cdot\alpha(wr(x)), u\rangle \\ &= \langle -\alpha((wr(x))r(\alpha(u))) + \alpha(wr(x))\cdot\alpha(u) - \alpha(\ell(\alpha(wr(x)))u), b^*\rangle, \\ \langle RHS, x\rangle &= \langle u, -(\alpha^*(b^*)\ell^*(\alpha(w)))\ell^*(x) - \alpha^*(R^*_{\hat{A}}(\alpha^*(b^*))w)\ell^*(x) + \alpha^*(b^*L^*_{\hat{A}}(\alpha(w)))\ell^*(x)\rangle \\ &= \langle -\alpha(\ell(\alpha(w))(\ell(x)u)), b^*\rangle - \langle \alpha(\ell(x)u)\cdot\alpha^*(b^*), w\rangle + \langle \alpha(w)\cdot\alpha(\ell(x)w), b^*\rangle \\ &= \langle -\alpha(\ell(\alpha(w))(\ell(x)u)) - \alpha(wr(\alpha(\ell(x)u))) + \alpha(w)\cdot\alpha(\ell(x)u), b^*\rangle. \end{aligned}$$

So Eq. (71) holds if and only if Eq. (67) holds. □

Corollary 4.8. *Let $(A, \cdot)$ be a $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension.*

(i) *Let $(R, \circ, \ell, r)$ be an A-bimodule $\mathbf{k}$-algebra with finite $\mathbf{k}$-dimension. Let $\alpha, \beta : R \to A$ be two linear maps such that α is an extended $\mathcal{O}$-operator of weight λ with modification β of mass (κ, μ), i.e., β is an A-bimodule homomorphism and the conditions* (20) *and* (21) *in Definition 2.9 hold, and α and β satisfy Eq.* (22)*. Then $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ is a skew-symmetric solution of the GAYBE* (59) *if and only if the following equations hold:*

$$\lambda\ell(\alpha(u\circ v))w = \lambda ur(\alpha(v\circ w)), \quad \forall u, v, w \in R, \tag{77}$$

$$\lambda\alpha(u(vr(x))) = \lambda\alpha(u\circ v)\cdot x, \quad \forall u, v \in R, x \in A, \tag{78}$$

$$\lambda\alpha((\ell(x)u)\circ v) = \lambda x\cdot\alpha(u\circ v), \quad \forall u, v \in R, x \in A. \tag{79}$$

In particular, when $\lambda = 0$, i.e., α is an extended $\mathcal{O}$-operator of weight zero with modification β of mass (κ, μ), $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ is a skew-symmetric solution of the GAYBE (59)*.*

(ii) *Let* $(R, \circ, \ell, r)$ *be an* A*-bimodule* $\mathbf{k}$*-algebra with finite* $\mathbf{k}$*-dimension. Let* $\alpha : R \to A$ *be an* $\mathcal{O}$*-operator of weight* λ*. Then* $\alpha - \alpha^{21}$ *identified as an element of* $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ *is a skew-symmetric solution of the GAYBE if and only if Eq. (77), Eq. (78) and Eq. (79) hold.*

(iii) *Let* (V, ℓ, r) *be a bimodule of* A *with finite* $\mathbf{k}$*-dimension. Let* $\alpha, \beta : V \to A$ *be two linear maps such that* α *is an extended* $\mathcal{O}$*-operator with modification* β *of mass* κ*. Then* $\alpha - \alpha^{21}$ *identified as an element of* $(A \ltimes_{r^*,\ell^*} V^*) \otimes (A \ltimes_{r^*,\ell^*} V^*)$ *is a skew-symmetric solution of the GAYBE.*

(iv) *Let* $\alpha : A \to A$ *be a linear endomorphism of* A*. Suppose that* α *satisfies Eq. (35). Then* $\alpha - \alpha^{21}$ *identified as an element of* $(A \ltimes_{R^*,L^*} A^*) \otimes (A \ltimes_{R^*,L^*} A^*)$ *is a skew-symmetric solution of the GAYBE.*

(v) *Let* $(R, \circ, \ell, r)$ *be an* A*-bimodule* $\mathbf{k}$*-algebra of finite* $\mathbf{k}$*-dimension. Let* $\alpha, \beta : R \to A$ *be two linear maps such that* α *is an extended* $\mathcal{O}$*-operator with modification* β *of mass* $(\kappa, \mu) = (0, \mu)$*, i.e.,* β *is an* A*-bimodule homomorphism and the condition (21) in Definition 2.9 holds, and* α *and* β *satisfy the following equation:*

$$\alpha(u) \cdot \alpha(v) - \alpha(\ell(\alpha(u))v + ur(\alpha(v))) = \mu\beta(u \circ v), \quad \forall u, v \in R.$$

Then $\alpha - \alpha^{21}$ *identified as an element of* $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ *is a skew-symmetric solution of the GAYBE.*

Proof. (i) Since α is an extended $\mathcal{O}$-operator of weight λ with modification β of mass (κ, μ), by Theorem 4.7, $\alpha - \alpha^{21}$ identified as an element of $(A \ltimes_{r^*,\ell^*} R^*) \otimes (A \ltimes_{r^*,\ell^*} R^*)$ is a skew-symmetric solution of the GAYBE (59) if and only if the following equations hold:

$$\begin{aligned} & -\lambda\ell(\alpha(u \circ v))w + \kappa\ell(\beta(u) \cdot \beta(v))w + \mu\ell(\beta(u \circ v))w \\ = \; & -\lambda ur(\alpha(v \circ w)) + \kappa ur(\beta(v) \cdot \beta(w)) + \mu ur(\beta(v \circ w)), \end{aligned} \tag{80}$$

$$\begin{aligned} & -\lambda\alpha((ur(x)) \circ v) + \kappa\beta(ur(x)) \cdot \beta(v) + \mu\beta((ur(x)) \circ v) \\ = \; & -\lambda\alpha(u \circ (l(x)v)) + \kappa\beta(u) \cdot \beta(\ell(x)v) + \mu\beta(u \circ (\ell(x)v)), \end{aligned} \tag{81}$$

$$\begin{aligned} & -\lambda\alpha(u \circ (vr(x))) + \kappa\beta(u) \cdot \beta(vr(x)) + \mu\beta(u \circ (vr(x))) \\ = \; & -\lambda\alpha(u \circ v) \cdot x + \kappa(\beta(u) \cdot \beta(v)) \cdot x + \mu\beta(u \circ v) \cdot x, \end{aligned} \tag{82}$$

$$\begin{aligned} & -\lambda\alpha((\ell(x)u) \circ v) + \kappa\beta(\ell(x)u) \cdot \beta(v) + \mu\beta((\ell(x)u) \circ v) \\ = \; & -\lambda x \cdot \alpha(u \circ v) + \kappa x \cdot (\beta(u) \cdot \beta(v)) + \mu x \cdot \beta(u \circ v), \end{aligned} \tag{83}$$

for any $u, v \in R, x \in A$. Since β is an A-bimodule homomorphism and the conditions (20) and (21) in Definition 2.9 hold, we have Eq. (77) holds if and only if Eq. (80) holds, Eq. (78) holds if and only if Eq. (82) holds, Eq. (79) holds if and only if Eq. (83) holds and Eq. (81) holds automatically.

(ii) This follows from Item (i) by setting $\kappa = \mu = 0$.

(iii) This follows from Item (i) by setting $\lambda = \mu = 0$.

(iv) This follows from Item (iii) for $(V, \ell, r) = (A, L, R)$ and $\beta = \mathrm{id}$.

(v) This follows from Item (i) by setting $\lambda = \kappa = 0$. ∎

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Chern Institute of Mathematics& LPMC, Nankai University, Tianjin 300071, P.R. China
E-mail address: baicm@nankai.edu.cn

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, U.S.A.
E-mail address: liguo@newark.rutgers.edu

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China
E-mail address: xiangn_math@yahoo.cn